Chapter 5

Annuities-Certain and their Value at Fixed Rate

5.1. General aspects

From now on we will consider problems that more frequently come into the financial practice, solving them in light of given theoretical formulation and on the basis of the *financial equivalence principle* following a prefixed exchange law.

The aforementioned principle was applied in Chapter 4 where, referring mostly to complex financial operations evaluated with exchange laws at *fixed rate* (i.e. constant in time; we can thus talk – as already mentioned – of *flat structure* rates, as in the regimes described in Chapter 3), their *values V(t)* and also *reserves M(t)* and *W(t)* are found at a generic time *t*. We stressed there the importance of fair operations, such that, if the exchange law is strongly decomposable, $\forall t$, we obtain *V(t) = W(t)-M(t) = 0*.

In this chapter we will consider the application of the correspondences in both sides between *flows* given by the operation and *funds* given by their capital values, *V(t)* at a given time *t*, all in a specific case: that of operation $\hat{O}$ constituted by a finite or infinite sequence of dated amounts *with the same sign*, that for one of the contracting parts is positive (and then they are incomes). Assuming, as it is used, an exchange law such that the equivalent amounts at different times always keep the same sign in the given temporal interval, the capital value *V(t)* of $\hat{O}$ at whichever time *t* has the same sign as the concordant transactions and therefore $\hat{O}$ can never be a fair operation, whichever exchange law parameters are used. However, a fair

operation $O*$ is obtained by adding to $\hat{O}$ the supply made by the opposite of such capital value paid at the evaluation time.

We will usually call an *annuity* the particular unfair operation $\hat{O}$, formed by a sequence of dated amounts with the same sign and made at equal intervals in time[1]. We will use the following definitions when referring to an annuity:

– *period* = constant temporal distance between two consecutive payments, usually of one year, or a multiple or sub-multiple of this;

– *frequency* = inverse of period, i.e. the number of payments per year;

– *interval* = time separating the beginning of the first period and the end of the last;

– *term* = length of the interval;

– *installment* = payment amount, constant or varying.

In addition, we will distinguish the annuities in the following ways:

– *annual*, when the period is one year, standard unit measure of time, or *fractional* or *pluriannual,* if the period is a submultiple or a multiple of one year;

– *annuity-due*, when the payment is made at the beginning of each period, or *annuity-immediate,* when it is made at the end of each period; a case of theoretical interest is that of *continuous annuity*, when the period tends towards zero and we have a continuous flow of payments;

– *certain*, if we assume that the established payments will be made with certainty, or *contingent*, if we assume that the payment of each installment is made only if a given event occurs[2]; in this part we will not consider *contingent* annuities and thus "certain" annuities will always be implied.

– *constant or varying*, referring to the installment sequence;

– *temporary or perpetuity*, if the term is finite or not.

EXAMPLE 5.1

1) The monthly payments for the rent of real estate can be considered as a certain annuity-due which is constant (or varying), monthly and temporary.

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1 Originally the meaning of the word "annuity" was restricted to annual payments, but it has been extended to include payments made at other regular intervals as well.
2 For a discussion on *contingent* annuities, which we consider in the field of "actuarial mathematics", it is necessary to have knowledge of basic probability calculus.

2) The future wages of a worker is a contingent annuity (considering the possibility of leaving the job due to death, invalidity, resignation, etc.), varying (due to the variation of wage), weekly- or monthly-immediate and temporary.

3) The "landed rent" of a cultivated field is, from an objective viewpoint (i.e. not considering the change of owners), a certain annuity, varying for perpetuity.

4) Another example of annuities is the payment of bills, accommodation expenses, etc.

In the problems of evaluation and negotiation we are interested in the *annuity capital value* even more than the annuity itself. This is, on the basis of what we have stated above, the amount that, associated with the evaluation time, gives rise to the indifferent supply[3] to the sequence of concordant supplies that form the annuity. The value thus depends on the evaluation time and the financial exchange law.

Usually this time is at the end of the annuity interval or at its initial time but it can even be before this. In the first case the capital value is called *final value or accumulated value*; in the second case *initial value* or *present value of a prompt annuity*; in the third case *present value of a delayed annuity*[4].

As concerns the exchange law, if the annuities are multi-year, a compound law, with a given interest conversion period and the corresponding rate per period, is usually used. For a short-term annuity, we usually use a simple interest law for the evaluation of the final value and a simple discount law for that of the initial value. Such financial laws are uniform, thus the annuity interval can always be translated, without changing the results[5].

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3 Or "equivalent" in the sense specified in Chapter 2, if the exchange law is strongly decomposable (s.dec.).

4 We usually distinguish between "annuity" and "delayed annuity" according to the comparison of their initial times and their evaluation times, something that does not take the payment characteristics into consideration. We observe that if we limit ourselves to the preceding choices, the evaluation time is never inside the annuity interval, so that in order to calculate the capital value, consideration of a complete exchange law, union of accumulation and, possibly conjugate, discount laws is not needed. In fact, it is enough to use a discount law for the present value and an accumulation law for the final value. Therefore, the weak decomposability of such laws is enough to obtain the equivalence between $\hat{O}$ and its final or present value at the evaluation time.

5 Sometimes a distinction between *simple annuity*, when the conversion and payment period coincide, and *general annuity*, when such periods do not coincide, is introduced; but they are usually commensurable (i.e. the ratio of their length is a rational number). Furthermore, a general annuity can always be led back to a simple annuity using equivalent rates to obtain a conversion with the same period as the annuity rates (see Hummel, Seebeck (1969)).

In Chapter 6, when discussing *amortization* and *accumulation*, we will consider in applicative terms the step, briefly considered above, from a unfair operation $\hat{O}$ of an *annuity* to a fair operation O^* associated with it. It will be enough to add to $\hat{O}$ a supply given by the couple of numbers: [a time extreme of the annuity interval; the opposite of the value at such time]. Due to the financial equivalence between the whole of the supplies of an annuity between T_1 and T_2 and its initial value in T_1 or final value in T_2, we can conclude that:

a) the annuity payments are installments of a debt *amortization* equal to their initial value $V(T_1)$, in the sense that if a loan of amount $V(T_1)$ has been made, the annuity supplies *amortize* the debt i.e. pay it back both for the principal and for the charged interest, if the discount law applied to the annuity corresponds to the law that rules the loan;

b) the annuity payments are installments for the *accumulation* of a capital (i.e. *funding*) equal to their final value $V(T_2)$, in the sense that, depositing the dated amounts of the annuity into a profitable account according to the applied accumulation law, such an account accumulates (considering also the accrued allowed interests) a credit that will reach in T_2 the value $V(T_2)$.

5.2. Evaluation of constant installment annuities in the compound regime

5.2.1. *Temporary annual annuity*

For simplicity, choosing as $t=0$ the beginning of the interval, let us now calculate the *initial value* (IV) at the annual rate i of a *temporary annual* annuity – thus featured in the interval $[0,n]$ by payments at the beginning or end of each year, according to the annuity being *due* or *immediate*, which is defined as the sum of the present values of each payment, and indicated with the symbol V_0 or $\ddot{V}_0$. Let n be the length of the annuity and thus the number of payments.

In the specific case of a *unitary* annual annuity (i.e. with unitary installments) which is *temporary*, or respectively *immediate* or *due*, for the IV we use the symbols $a_{\overline{n}|i}$ or $\ddot{a}_{\overline{n}|i}$ [6], referring to annual periods and rates, and by definition

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6 Such symbols, separately for immediate and due case, depend on the duration (or number of periods) n and the per period equivalent rate i. The diaeresis denotes annuity-due. The results of suitable calculations of these values for the immediate case (those for the due case can be calculated using the previous case: see e.g. (5.2)) and of other quantities are scheduled in specific "financial tables", depending on the most important parameters. However, the increasing availability of very good pocket scientific calculators enables exact calculations of the value of any parameters, thus making tables obsolete.

$$a_{\overline{n}|i} := [(1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-n}] \tag{5.1}$$

$$\ddot{a}_{\overline{n}|i} := [1 + (1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-(n-1)}] = (1+i)\ a_{\overline{n}|i} = 1 + a_{n-\overline{1}|i}$$

According to the equivalence principle, it is immediately verified that, as $v = (1+i)^{-1}$ = annual discount factor, $d = 1-v = iv$ = annual discount rate, we obtain[7]:

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} = \frac{1-v^n}{i} \ ; \ \ddot{a}_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{d} = \frac{1-v^n}{d} \tag{5.2}$$

and thus, for the annuity-immediate with installment R and annuity-due with installment $\ddot{R}$, the IV are respectively:

$$V_0 = R\ a_{\overline{n}|i} \ ; \ \ddot{V}_0 = \ddot{R}\ \ddot{a}_{\overline{n}|i} \tag{5.3}$$

Due to the observation at the end of section 5.1, in (5.3), R is the *constant installment of delayed amortization* in n years at the annual rate i of the debt V_0, ($\ddot{R}$ is for the *advance* amortization of the debt $\ddot{V}_0$).

Exercise 5.1

Calculate the amount to be paid today as an alternative to 5 payments of €1,000 with a deadline at the end of each year, with the annuity starting today, and adopting a compound annual exchange law at the annual delayed interest rate of 8.25%

A. We apply (5.3) using: $R = 1,000$; $i = 0.0825$; $n = 5$.

The following is obtained

$$V_0 = 1,000\ (1 - 1.0825^{-5})/0.0825 = 3,966.54$$

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7 This and the following formulations can be proved algebraically, but we prefer to use financial arguments, as we consider them to be more appropriate here. Thus, to obtain (5.2), recalling that it is indifferent to defer an income if in the meantime the interest is accrued according to the prefixed accumulation law, using the compound regime and valuing at time 0, it is indifferent to receive the amount S at time 0 (present value = S) or receiving it at time n (present value = $S\ v^n$) with the addition of the delayed annual interests, forming an annual annuity-immediate for n years of installment Si (present value = $Si\ a_{\overline{n}|i}$) or advance, forming an annual annuity-due for n years of installment Sd (present value = $Sd\ \ddot{a}_{\overline{n}|i}$). Thus, $\forall S$, the financial equivalences: $S = S\ v^n + Si\ a_{\overline{n}|i}$; $S = S\ v^n + Sd\ \ddot{a}_{\overline{n}|i}$, and thus (5.2) can be obtained.

If we are interested in the *final value (FV)* V_n (or $\ddot{V}_n$) of the *annual temporary annuity-immediate or -due*, defined as the sum of the accumulated values in n of each payment, due to the decomposability of the compound law, it is equivalent to accumulating each payment until time n and adding the results or discount each payment until time 0 and accumulating for n years the sum of the obtained values. Therefore,

$$V_n \;=\; (1+i)^n\, V_0 \qquad ; \qquad \ddot{V}_n \;=\; (1+i)^n\, \ddot{V}_0 \tag{5.4}$$

Therefore, indicating with $s_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i}$ the *final value* of *unitary temporary annual annuity*, respectively *-immediate* and *-due*, and using the same argument as for the IV, by definition the following is the result:

$$s_{\overline{n}|i} := 1 + (1+i) + (1+i)^2 + \dots + (1+i)^{n-1} \tag{5.5}$$

$$\ddot{s}_{\overline{n}|i} := (1+i) + (1+i)^2 + \dots + (1+i)^n$$

and the following is easily obtained[8]

$$s_{\overline{n}|i} = (1+i)^n\, a_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \tag{5.6}$$

$$\ddot{s}_{\overline{n}|i} = s_{\overline{n+1}|i} - 1 = \ddot{a}_{\overline{n}|i}\,(1+i)^n = \frac{(1+i)^n - 1}{d}\, s_{\overline{n}|i}$$

while for annuity-immediate with installment R or -due with installment $\ddot{R}$ it results in

$$V_n = R\, s_{\overline{n}|i} \qquad ; \qquad \ddot{V}_n = \ddot{R}\, \ddot{s}_{\overline{n}|i} \tag{5.7}$$

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8 The last terms of (5.6) can be obtained from financial equivalence valuing at time n the amount S paid in 0 or the same amount paid in n plus the annual delayed or advance interest, between 0 and n and obtaining the equalities: $S\,(1+i)^n = S + Si\, s_{\overline{n}|i}$; $S\,(1+i)^n = S + Sd\, \ddot{s}_{\overline{n}|i}$.

In (5.3) R is the *constant delayed installment of the accumulation* of V_n in n years at the annual rate i whereas $\ddot{R}$ is the *constant advance installment of the accumulation* of $\ddot{V}_n$.

The following symbols are frequently used and can be found on tables for the most common values of n and i:

$$\alpha_{\overline{n}|i} = 1/a_{\overline{n}|i} \; ; \; \ddot{\alpha}_{\overline{n}|i} = 1/\ddot{a}_{\overline{n}|i} \; ; \; \sigma_{\overline{n}|i} = 1/s_{\overline{n}|i} \; ; \; \ddot{\sigma}_{\overline{n}|i} = 1/\ddot{s}_{\overline{n}|i} . \qquad (5.8)$$

These form the coefficient to be applied to the IV or FV of an annuity-immediate or annuity-due to obtain the constant installments. In fact, obtaining R and $\ddot{R}$ from (5.3) and (5.7), it follows that

$$R = V_0 \, \alpha_{\overline{n}|i} \; ; \; \ddot{R} = \ddot{V}_0 \, \ddot{\alpha}_{\overline{n}|i} \; ; R = V_n \, \sigma_{\overline{n}|i} \; ; \; \ddot{R} = \ddot{V}_n \, \ddot{\sigma}_{\overline{n}|i} \qquad (5.8')$$

The values in (5.8') thus give the amortization installment of the debt V_0 and the delayed or advance funding installment of the capital V_n[9].

Calculation of rate and length

Considering only the annuity-immediate case, (5.3) is a constraint between the quantities V_0, R, n, i, which enables expression one to be dependent on the other three. The first parts of (5.3) and (5.8) explain V_0 and R. The calculation of i is reduced to that of the IRR (see Chapter 4) of the operation $O* = \hat{O}\sqcup(0,-V_0)$ defined in section 5.1. Sometimes more needs to be said about the calculation of the *implicit length n*.

From the 1st part of (5.3) we obtain, recalling (5.2):

$$\frac{V_0}{R} = \frac{1-(1+i)^{-n}}{i} \quad \text{i.e.} \quad 1 - \frac{iV_0}{R} = (1+i)^{-n} = e^{-\delta n}$$

Considering the natural logarithm and (3.30') we obtain the implicit length:

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9 The comparison between (5.8) and (5.8') makes it possible to give a financial meaning to the values $\alpha_{\overline{n}|i}$, $\ddot{\alpha}_{\overline{n}|i}$, $\sigma_{\overline{n}|i}$, $\ddot{\sigma}_{\overline{n}|i}$. They are, in order, the constant delayed and advance installments of amortization of the unitary debt and of funding of the unitary capital.

$$n = \frac{-\ln\left(1 - \dfrac{iV_0}{R}\right)}{\delta} \tag{5.8''}$$

Solution (5.8'') is positive, because $0 < 1 - iV_0/R < 1$, but usually it is not a natural number. We can consider the natural n_0 that better approximates the solution, again calculating (if desired) the IV as a function of n_0.

Between the quantities (5.8) there are the following relations, which have a relevant financial meaning[10]:

$$\alpha_{\overline{n}|i} = \sigma_{\overline{n}|i} + i \; ; \; \ddot{\alpha}_{\overline{n}|i} = \ddot{\sigma}_{\overline{n}|i} + d \tag{5.9}$$

It is useful to consider that (as can be deduced from their algebraic values and financial meaning):

– $a_{\overline{n}|i}$ is an increasing function of n and decreasing of i;

– $\alpha_{\overline{n}|i}$ is a decreasing function of n and increasing of i;

– $s_{\overline{n}|i}$ is an increasing function of n and increasing of i;

– $\sigma_{\overline{n}|i}$ is a decreasing function of n and decreasing of i.

The same dynamics apply to the annuity-due values.

We have examined, so far, the evaluation of annuities carried out at the beginning of the interval (and thus, as already specified in section 5.1, we talk about IV and prompt annuities). Furthermore, we obtain *present values of delayed annuity* (PVDA) if the evaluation time precedes the beginning of the interval. Putting it in -r we have a deferment, and then an increment, of the discount times of all payments, of r years (r can also be not integer). Therefore, indicating with $_{r/}a_{\overline{n}|i}$ or $_{r/}\ddot{a}_{\overline{n}|i}$ the PVDA in the case of unitary temporary installments, annuity-immediate or annuity-due, it is obvious that

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10 Equation (5.9) can be easily deduced algebraically but it can be justified financially with an equivalence between amortizations: for the debtor it is equivalent to paying, at the end or beginning of the year, to the creditor the constant amortization installment referred to as the unitary debt (for n years at rate i) or to paying only the interest amount, i or d, and accumulating in n years the unitary capital to pay back to the creditor in only one payment at the end. In the alternative, the annual constant payments must be equal. Thus, equation (5.9) is justified. We will come back to this in Chapter 6 when considering "American" amortization.

$$_{r/}a_{\overline{n}|i} = v^r\, a_{\overline{n}|i} = \frac{v^r - v^{r+n}}{i}\; ;\; _{r/}\ddot{a}_{\overline{n}|i} = v^r\, \ddot{a}_{\overline{n}|i} = \frac{v^r - v^{r+n}}{d} \tag{5.10}$$

and in the case of constant installment R of an annuity-immediate or $\ddot{R}$ of an annuity-due, we obtain:

$$_{r/}V_0 = R\, _{r/}a_{\overline{n}|i}\; ;\; _{r/}\ddot{V}_0 = \ddot{R}\, _{r/}\ddot{a}_{\overline{n}|i} \tag{5.3'}$$

5.2.2. *Annual perpetuity*

Let us now consider *annual perpetuity*, observing that, in the case of constant installments, the FV is not considered because it goes to infinity[11]. However, the IV are finite and are obtained using $n \to +\infty$ in the previous formulae from (5.1) to (5.3). It follows that[12]

$$a_{\overline{\infty}|i} = 1/i,\; \ddot{a}_{\overline{\infty}|i} = 1/d\; ;\; V_0 = R/i,\; \ddot{V}_0 = \ddot{R}/d \tag{5.11}$$

and, for the PVDA:

$$_{r/}a_{\overline{\infty}|i} = v^r/i;\; _{r/}\ddot{a}_{\overline{\infty}|i} = v^r/d;\; _{r/}V_0 = R\, v^r/i;\; _{r/}\ddot{V}_0 = \ddot{R}\, v^r/d \tag{5.11'}$$

Exercise 5.2

1) Calculate the initial and final value of an annual annuity-due with constant installment $\ddot{R} = \text{€}150$, annual interest rate $i = 8.55\% = 0.0855$, length $n = 17$.

A. Using (5.3) we obtain

$$\ddot{V}_0 = 150 + 150\, (1 - 1.0855^{-16})/0.0855 = 150\, (1 + 8.5484723) = 1{,}432.27$$

In addition, we obtain $d = 0.0855/1.0855 = 0.0787656$ and, applying (5.6),

$$\ddot{V}_n = \ddot{R}\, \ddot{s}_{\overline{n}|i} = (1.0855^{17} - 1)/0.07876555 = (1.0855^{18} - 1)/0.0855 = 38.515980$$

and thus

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11 It should be calculated in the compound regime, at whatever non-negative rate, as the sum of the elements of a geometric sequence with ratio ≥ 1, which is positively diverging.
12 The expressions in (5.11), with their formal simplicity, are of fundamental importance in the accumulation problems of perpetual incomes of lasting assets.

$$\ddot{V}_{17} = 150 \cdot 38.5159796 = 5{,}777.40$$

(5.4) is soon verified, resulting in

$$(1.0855)^{17} 1432.27 = 4.033732164\ 1432.27 = 5{,}777.40$$

2) An estate can be bought with an advance of €5,600 and a loan that involves 15 annual delayed installments of €850 each. Calculate the equivalent price in cash, if the annual loan rate is 6%.

A. Let P be such a price, and applying (5.2) the result is:

$$P = 5600 + 850\ (1\text{-}1.06^{-15})/0.06 = 5600 + 8255.41 = 13{,}855.41.$$

3) A reserve fund of a company at the closing balance is €156,500. If we want to increase it in 5 years to the level of €420,000 through constant earmarking at the end of each following year in a savings account at the compound annual interest of 6%, calculate the amount of each annual earmarking, assuming they are constant.

A. Denoting the earmarking by C, it is given by

$$C = (420{,}000 - 156{,}500)\,\sigma_{5|0.06} = 263{,}500\ 0.06/(1.06^5 - 1) = 263{,}500 \cdot 0.17739640$$
$$= €41{,}743$$

4) Verify (5.9) for the values $n = 15$, $i = 0.09$

A. The first formula gives rise to the equality: $0.12405888 = 0.03405888 + 0.09$ and the second to: $0.11381549 = 0.03124668 + 0.08256881$, obtained from the previous one multiplying by $v=1/(1 + i) = 1/1.09 = 0.91743119$.

5.2.3. *Fractional and pluriannual annuities*

In sections 5.2.1 and 5.2.2 we considered the evaluation in the compound regime, with annual conversion of interest, of annuities with annual installments. The same formulae can be used for *m-fractional annuities* i.e. with installments of frequency m (usually $m = 2, 3, 4, 6, 12, 52, 360$ for the usual fraction of a year, even if only the constraint $m\text{-}1 \in \mathcal{N}$ follows from the definition) for the evaluation of which we use the m-fractional conversion of interest and then it is given the delayed per period rate $i_{1/m}$ for $1/m$ of a year or the intensity $j(m) = m\ i_{1/m}$. It is sufficient to use in such formulae $i_{1/m}$ instead of i and as a temporal parameter the number of payments, then changing the unit measure of time. For the *pluriannual annuities*

with a payment every p years it is sufficient to use $m = 1/p$ and instead of i the p-annual equivalent rate, given by $i_p = (1+i)^p - 1$ [13].

In fact, we recall that in a compound accumulation process we obtain the same return with an annual conversion at rate i or with the m-fractional conversion at the rate per period $i_{1/m}$ if the two rates are equivalent, i.e. linked by (3.26).

Temporary fractional annuities

The IV at the delayed rate i of an *m-fractional annuity with length* $n \leq +\infty$ (i.e. an annuity such that the annual amount R is fractionated into the annual interval in m equally spaced installments, the amount of which is $R_{1/m} = R/m$, so the annuity has period $1/m$) can be evaluated, distinguishing the annuity-immediate from the annuity-due case, through formulae analogous to (5.3), obtaining

$$V_0^{(m)} = R_{1/m} a_{m\overline{n}|i_{1/m}} \; ; \; \ddot{V}_0^{(m)} = R_{1/m} \ddot{a}_{m\overline{n}|i_{1/m}} \qquad [14] \qquad (5.12)$$

In the specific case of an annuity *unitary m-fractional* (i.e. with installments of amount $1/m$ so as to have a unitary annual amount) *temporary*, respectively *annuity-immediate* or *annuity-due,* for the IV we use the symbols $a_{\overline{n}|i}^{(m)}$ or $\ddot{a}_{\overline{n}|i}^{(m)}$ and, analogously to (5.2), we obtain the following formulae

$$a_{n|i}^{(m)} = \frac{1-v^n}{j(m)}; \; \ddot{a}_{n|i}^{(m)} = (1+i)^{1/m} a_{n|i}^{(m)} = \frac{1-v^n}{\rho(m)} \quad [15] \qquad (5.13)$$

In general, with an annual total R, (5.12) can be rewritten, using (5.13), as

$$V_0^{(m)} = R_{1/m} a_{m\overline{n}|i_{1/m}} \; ; \; \ddot{V}_0^{(m)} = R_{1/m} \ddot{a}_{m\overline{n}|i_{1/m}} \qquad (5.12')$$

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13 Given that these transformations leave the period of the annuity and the conversion unchanged, we are still in the case of basic annuities, specified in footnote 5.

14 In the fractional annuity the number of payments nm can be large and, if we use financial tables, it can be higher than the maximum in the table. In such cases, the following decomposition can be useful: $a_{n+\overline{p}|i} = a_{\overline{n}|i} + (1+i)^{-n} a_{\overline{p}|i}$, which enables calculation of the 1st member when the length $n+p$ goes beyond the limit of the table, provided that $(n.p)$ is whichever duration included in the table.

15 Proceeding analogously to footnote 7, (5.13) can be obtained taking into account the financial equivalence between the amount S in 0 or the same amount in n adding the delayed or advance interest paid with frequency m. Their annual total is thus, respectively $mSi_{1/m}$ or $mSd_{1/m}$. Therefore, the equivalences give rise to the equations: $S = S j(m) a_{\overline{n}|i}^{(m)} + S (1+i)^{-n}$, $S = S \rho(m) a_{\overline{n}|i}^{(m)} + S (1+i)^{-n}$, and thus (5.13).

The step from (5.12) to (5.12') is justified observing that

$$V_0^{(m)} = \frac{R}{m}\frac{1-(1+i_{1/m})^{-mn}}{i_{1/m}} = R\frac{1-(1+i)^{-n}}{j(m)}$$

$$\ddot{V}_0^{(m)} = \frac{R}{m}\frac{1-(1+i_{1/m})^{-mn}}{i_{1/m}}(1+i_{1/m}) = R\frac{1-(1+i)^{-n}}{\rho(m)}$$

To obtain the FV of a temporary m-fractional annuity for n years, annuity-immediate or annuity-due, given the decomposability of the compound laws it is enough to accumulate the IV during the interval of the annuity. If the annual total is unitary, the FV, indicated in the two cases with $s_{n|i}^{(m)}$ and $\ddot{s}_{n|i}^{(m)}$, are

$$s_{n|i}^{(m)} = (1+i)^n\ a_{n|i}^{(m)}\ ;\ \ddot{s}_{n|i}^{(m)} = (1+i)^n\ \ddot{a}_{n|i}^{(m)} \tag{5.6'}$$

In general, with annual total R, analogously to (5.4'), the FV are given by

$$V_n^{(m)} = R\ s_{n|i}^{(m)} = R\ (1+i)^n\ a_{n|i}^{(m)} = (1+i)^n\ V_0^{(m)} \tag{5.4'}$$
$$\ddot{V}_n^{(m)} = R\ \ddot{s}_{n|i}^{(m)} = R\ (1+i)^n\ \ddot{a}_{n|i}^{(m)} = (1+i)^n\ \ddot{V}_0^{(m)}$$

Exercise 5.3

Calculate the IV and FV of the annuity formed by the income flow with monthly delayed installment of €650 for 10 years at the nominal annual rate 12-convertible of 9%.

A. We have $i_{1/12} = 0.0075$, $i = 0.0938069$, $R = 7,800$, and then

$$V_0^{(12)} = 650\ \frac{1-1.0075^{-120}}{0.0075} = 650 \cdot 78.9416927 = 51,312.10\ , \text{ or}$$

$$V_0^{(12)} = 7,800\ a_{10|i}^{(12)} = 7,800\frac{1-1.0938069^{-10}}{0.09} = 7,800 \cdot 6.5784744 = 51,312.10$$

In addition: $V_n^{(12)} = 5,1312.10 \cdot 1.0075^{120} = 125,784.28$

Applying footnote 14 and assuming there is a financial table with a maximum length of 100, choosing the length 70 and 50, with 70+50=120, this results in

$$a_{120|0.0075} = a_{70|0.0075} + 1.0075^{-70} a_{50|0.0075} = 54.3046221 +$$
$$0.59271533 \cdot 41.5664471 = 78.9416925,$$

and thus $V_0^{(12)} = 51312.10$, i.e. the same value as previously.

If, with the same data, the installments are in advance, this results in: $d_{1/12} = 0.0074442$; $\rho(12) = 0.08933$ and

$$\ddot{V}_0^{(12)} = 7,800 \; \ddot{a}_{10|0.0938069}^{(12)} = 7,800 \; \frac{1 - 1.0938069^{-10}}{0.08933} = 7,800 \cdot 6.6278135 = 51,696.96$$

$$\ddot{V}_{10}^{(12)} = 51696.96 \cdot 1.0075^{120} = 126,727.71$$

For completeness, let us mention briefly annuities *m-fractional delayed* for *r* years, for which the PVDA are obtained multiplying by v^r the corresponding IV With a unitary annual total we have, with the obvious meaning of the symbols

$$_{r/}a_{n|i}^{(m)} = v^r a_{\overline{n}|i}^{(m)} = \frac{v^r - v^{n+r}}{j(m)}; \; _{r/}\ddot{a}_{n|i}^{(m)} = (1+i)^{1/m} \, _{r/}a_{n|i}^{(m)} = \frac{v^r - v^{n+r}}{\rho(m)} \quad (5.13')$$

while in general, with installment *R/m* it is enough to multiply by *R* the values (5.13').

Exercise 5.4

Using the data in exercise 5.3, calculate the PVDA with 4 years deferment.

A. The following is obtained:

$$_{4/}a_{10|9.38069\%}^{(12)} = 1.0938069^{-4} \, a_{10|9.38069\%}^{(12)} = 0.6986141 \cdot 6.5784744 = 4.5958166$$

$$_{4/}\ddot{a}_{10|9.38069\%}^{(12)} = 1.0938069^{-4} \, \ddot{a}_{10|9.38069\%}^{(12)} = 0.6986141 \cdot 6.6278135 = 4.6302842$$

Fractional perpetuity

The IV of the *fractional perpetuity* are obtained from those for temporary values putting $n \to +\infty$ and taking into account that in such a case $v^n \to 0$. If the annual total is unitary, we obtain, analogously to (5.11),

$$a_{\infty|i}^{(m)} = 1/j(m) \; ; \; \ddot{a}_{\infty|i}^{(m)} = (1+i)^{1/m} \, a_{\infty|i}^{(m)} = 1/\rho(m) \qquad (5.14)$$

while with installment R/m

$$V_0^{(m)} = R \, a_{\infty|i}^{(m)} = R/j(m) \; ; \; \ddot{V}_0^{(m)} = R \, \ddot{a}_{\infty|i}^{(m)} = R/\rho(m) \qquad (5.14')$$

and, for the PVDA the result is[16]

$$_{r/}a_{\infty|i}^{(m)} = v^r/j(m) \quad ; \quad _{r/}\ddot{a}_{\infty|i}^{(m)} = v^r/\rho(m) \qquad (5.15)$$

$$_{r/}V_0^{(m)} = R \, v^r/ \, j(m) \quad ; \quad _{r/}\ddot{V}_0^{(m)} = \ddot{R} \, v^r/\rho(m)$$

Exercise 5.5

Using the data from Exercise 5.4, calculate the IV and the PVDA of an immediate or due perpetuity with flow equal to €5,600/year.

A. Applying (5.14') for the IV we obtain

$$V_0^{(m)} = 5,600 \, a_{\infty|9.38069\%}^{(12)} = \frac{5,600}{0.09} = 62,222.22$$

$$\ddot{V}_0^{(m)} = 5,600 \, \ddot{a}_{\infty|9.38069\%}^{(12)} = \frac{5,600}{0.08933} = 62,688.91;$$

and for the PVDA discounting we obtain

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16 A comparison between the formulae shows the intuitive fact that both for annual annuity, a fractional annuity and (as we will see) a continuous annuity, the following decomposition holds: the *IV* of a perpetuity is the sum of the *IV* of a corresponding temporary annuity and of the *PVDA* of the corresponding delayed annuity at the end of the previous one. In the simplest case, of a unitary annual annuity-immediate, the result is: $a_{\infty|i} = a_{\bar{r}|i} + {}_{r/}a_{\infty|i}$, following the identity: $1/i = (1-v^r)/i + v^r/i$. This splitting up is similar to the juridical splitting with usufruct and bare ownership (but in a different meaning as used in Chapter 4). The usufruct is like the temporary annuity, whereas the bare ownership is like the delayed annuity, which starts after the end of the temporary one. However – unlike what occurs in annuities-certain – the splitting up with usufruct and bare ownership leads to uncertain values, since it is linked to a random usufructuary lifetime. Therefore, in a random case we calculate mean values according to expected lifetime.

$$_{4/}V_0^{(12)} = 0.698614 \cdot 62222.22 = 43{,}469.32;$$

$$_{4/}\ddot{V}_0^{(12)} = 0.698614 \cdot 62688.91 = 43{,}795.35$$

or applying (5.15)

$$_{4/}V_0^{(12)} = 5{,}600 \, \frac{0.698614}{0.09} = 43469.32; \quad _{4/}V_0^{(12)} = 5{,}600 \, \frac{0.698614}{0.08933} = 43{,}795.35$$

Continuous annuities

Let us briefly consider *continuous annuities (temporary or perpetuities, prompt or delayed)*, characterized by a continuous flow of payments, which we assume here to be constant. They can be considered as a specific case of fractional annuities, for $m \to +\infty$. For uniformity of symbols and easier comparison we assume the flow of R per year, where R is also the amount paid in one year. Since the period goes to 0, the distinction between annuity-immediate and annuity-due does not make sense.

Indicating with $a_{n|i}^{(\infty)}$ the IV, with $s_{n|i}^{(\infty)}$ the FV (only if $n<\infty$), with $_{r/}a_{n|i}^{(\infty)}$ the PVDA of the *unitary* annuity ($R=1$), taking into account (5.13), (5.13'), (5.14), (5.15) and the convergences $\rho(m) \to \delta \leftarrow j(m)$ when $m \to +\infty$, using these limits the following is easily obtained:

$$a_{\overline{n}|i}^{(\infty)} = \frac{1-v^n}{\delta}; \; s_{\overline{n}|i}^{(\infty)} = \frac{(1+i)^n -1}{\delta}; \; _{r/}a_{\overline{n}|i}^{(\infty)} = \frac{v^r - v^{r+n}}{\delta}; \tag{5.16}$$

$$a_{\infty|i}^{(\infty)} = \frac{1}{\delta}; \; _{r/}a_{\infty|i}^{(\infty)} = \frac{v^r}{\delta}$$

Exercise 5.6

Using the data in exercise 5.4, calculate the values in (5.16) of the unitary perpetuities.

A. We have: $i = 0.0938069$ and $\delta = \ln 1.0938069 = 0.0896642$; thus the following is obtained:

$$a_{10|i}^{(\infty)} = 6.603113; \; s_{10|i}^{(\infty)} = 16.186588; \; _{4/}a_{10|i}^{(\infty)} = 4.613027;$$

$$a_{\infty|i}^{(\infty)} = 11.152723; \; _{4/}a_{\infty|i}^{(\infty)} = 7.791450.$$

If the annuity has flow *R,* it is enough to multiply by *R* the values of (5.16)[17].

Pluriannual annuity, temporary or perpetuities

We will now comment briefly on *pluriannual annuities*, characterized by constant installments, annuity-immediate or annuity-due, equally spaced over *p* years, therefore with frequency $1/p$. They find application, for example, in the evaluation of the charges due to industrial equipment renewal.

If such annuities are temporary, it is necessary that $n = kp$ (where $k \in \mathcal{N}$ is the number of installments). Thus, indicating with $i_p = (1+i)^p - 1 = i\, s_{\overline{p}|i}$ the *p*-annual rate equivalent to *i*, to obtain the capital value it is enough to assume an interval of *p* years as the new unit measure and apply the formulae for the annual annuities using $i_{1/p}$ as the rate and *k* as the length.

In more detail, considering as *unitary* (referring to the annual amount) *p-annual annuity* that with installment $R_p = p$, we indicate with $a_{\overline{n}|i}^{(1/p)}$ or $\ddot{a}_{\overline{n}|i}^{(1/p)}$ the IV of the *temporary* one with length *n*, *-immediate* or *-due*. It thus follows that:

$$a_{\overline{n}|i}^{(1/p)} = p\,\frac{1-(1+i_p)^{-k}}{i_p} = p\,\frac{1-(1+i)^{-n}}{(1+i)^p - 1} = p\,\sigma_{\overline{p}|i} < a_{\overline{n}|i} \qquad (5.17)$$

$$\ddot{a}_{\overline{n}|i}^{(1/p)} = (1+i)^p\, a_{\overline{n}|i}^{(1/p)} = p\,\frac{1-(1+i)^{-n}}{1-(1+i)^{-p}} = p\, a_{\overline{n}|i}\, \alpha_{\overline{p}|i} > a_{\overline{n}|i}$$

---

17 The value of continuous annuity can be calculated analytically in the compound regime using the integrals of continuous flow, which are discounted or accumulated. If the flow is the constant *R*, the results are:

$$R\,a_{\overline{n}|i}^{(\infty)} = R\,\frac{1-v^n}{\delta}\;;\quad R\,s_{\overline{n}|i}^{(\infty)} = \int_0^n R\,e^{\delta(n-t)}\,dt = R\,\frac{e^{\delta n}-1}{\delta}\;;$$

$$R_{r/}a_{\overline{n}|i}^{(\infty)} = \int_0^n R\,e^{-\delta t}\,dt = \int_r^{r+n} R\,e^{-\delta t}\,dt = R\,\frac{e^{-\delta r}-e^{-\delta(r+n)}}{\delta}\;;$$

$$R\,a_{\overline{\infty}|i}^{(\infty)} = \int_0^{+\infty} R\,e^{-\delta t}\,dt = \frac{R}{\delta}\;;\quad R_{r/}a_{\overline{\infty}|i}^{(\infty)} = \int_r^{+\infty} R\,e^{-\delta t}\,dt = R\,\frac{e^{-\delta r}}{\delta}\,,$$

i.e. the values obtained from (5.16). With varying flow $\varphi(t)$, the *IV* of a temporary annuity is given by $\int_0^n \varphi(t)\,e^{-\delta t}\,dt$, with obvious modification for the other cases.

and, in general, the IV of the analogous annuity with installment R_p-*immediate* or R_p-*due* are

$$V_0^{(1/p)} = R_p \, \frac{1-(1+i)^{-n}}{(1+i)^p - 1} = R_p \, a_{\overline{n}|i} \, \sigma_{\overline{p}|i}; \tag{5.17'}$$

$$\ddot{V}_0^{(1/p)} = R_p \, \frac{1-(1+i)^{-n}}{1-(1+i)^{-p}} = R_p \, a_{\overline{n}|i} \, \alpha_{\overline{p}|i}$$

Multiplying (5.17') by $(1+i)^n$ the FV of such annuities are obtained. Multiplying them instead by $(1+i)^{-r}$ the PVDA of the analogous annuity *p-annual temporary delayed* for *r* years are obtained. Using instead $n \to +\infty$, the IV of the analogous *p-annual perpetuity* are obtained. For the unitary IV it is found that

$$a_{\infty|i}^{(1/p)} = \frac{p}{(1+i)^p - 1} = \frac{1}{j(1/p)} \tag{5.18}$$

$$\ddot{a}_{\infty|i}^{(1/p)} = \frac{p}{1-(1+i)^p} = \frac{1}{\rho(1/p)} = a_{\infty|i}^{(1/p)} + p \quad {}^{18}$$

and it is generally sufficient to substitute R_p to p, obtaining:

$$V_0^{(1/p)} = \frac{R_p}{(1+i)^p - 1} = R_p \, a_{\infty|i} \, \sigma_{\overline{p}|i} \tag{5.18'}$$

$$\ddot{V}_0^{(1/p)} = \frac{R_p}{1-(1+i)^{-p}} = R_p \, a_{\infty|i} \, \alpha_{\overline{p}|i}$$

---

18 Observe that, as for (5.14), the values in (5.18) represent the reciprocal of the intensity per period on a length of *p* years. In the second part of (5.18), the last term is justified noting that the annuity-due value is obtained from the annuity-immediate value adding the initial $R_{1/p} = p$ and subtracting nothing due to perpetuity.

and also: $\ddot{V}_0^{(1/p)} = V_0^{(1/p)} + R$.

Exercise 5.7

1) Calculate the IV of a five-yearly annuity at the annual rate of 7%, -immediate or -due, with a length of 20 years or perpetuity, with delayed installments of €58,500.

A. By applying formulae from (5.17) to (5.18') the following is obtained:

– in the temporary case:

$$a_{\overline{20}|0.07}^{(1/5)} = 5 \, \frac{1-(1.07)^{-20}}{(1.07)^5 - 1} = 9.2110032$$

$$V_0^{(1/5)} = 58,500 \, \frac{1-(1.07)^{-20}}{(1.07)^5 - 1} = 107,768.74$$

$$\ddot{a}_{\overline{20}|0.07}^{(1/5)} = 5 \, \frac{1-(1.07)^{-20}}{1-(1.07)^{-5}} = 12.9189075$$

$$\ddot{V}_0^{(1/5)} = 58,500 \, \frac{1-(1.07)^{-20}}{1-(1.07)^{-5}} = 151,151.22$$

– in the perpetual case:

$$a_{\overline{\infty}|0,07}^{(1/5)} = \frac{5}{(1,07)^5 - 1} = 12.4207639 \quad ; \quad V_0^{(1/5)} = \frac{58500}{(1.07)^5 - 1} = 145,322.92$$

$$\ddot{a}_{\overline{\infty}|0,07}^{(1/5)} = \frac{5}{1-(1,07)^{-5}} = 17.9189075 \quad ; \quad \ddot{V}_0^{(1/5)} = \frac{58500}{(1.07)^5 - 1} = 203,822.92$$

2) For the functioning of a company the owner buys equipment that must be replaced, due to wear and obsolescence, every 5 years. A horizon of 20 years is established for the company's activity, for which the return rate is 7.45%. The mean cost of the equipment is evaluated in €255,000. On the basis of such estimations, calculate the IV for the purchase expenses of such equipment for the whole time horizon.

A. The IV asked for is that of a 5-year temporary annuity-due for 20 years, and according to (5.17') it is

$$\ddot{V}_0^{(1/5)} = \ddot{a}_{\overline{20}|0.0745}^{(1/5)} = 255,000 \, \frac{1-(1.0745)^{-20}}{1-(1.0745)^{-5}} = 255,000 \cdot 2.5259699 = 644,122.34$$

3) A forestry company wants to buy a wood, the income of which follows the periodic cutting down of the trees and use of the wood, with spontaneous reforestation, at a price which, according to the principle of "capitalization of income", is given by the present value, at a rate corresponding to cost-opportunity, of future profits. Supposing that:

– the trees are cut down every 12 years;

– costs and returns between the periods are compensated;

– the profit due to each harvest is €55,000;

– the evaluation rate is 6.20%;

calculate the price offered by the company, in the alternative hypotheses that

a) the trees have only just been cut down;

b) all the trees have an age of 7 years;

c) all the trees have an age of 12 years.

A. In the given problem the return can be considered perpetual[19]. Therefore, the price P following the accumulation of the profit (then the IV of the annuity of future returns) is obtained as follows:

a) $P = P_a$ is the IV of the 12-year annuity-immediate with constant installment $R_p = 55,000$:

$$P_a = \frac{R_{12}}{i_{12}} = \frac{55000}{1.062^{12} - 1} = 51,973.51$$

b) $P = P_b$ is the PVDA of a 12-year annuity-due delayed for 5 years, with constant installment $R_p = 55,000$:

$$P_b = (1+i)^{-5} \frac{R_{12}\,(1+i_{12})}{i_{12}} = \frac{55000 \cdot 1.062^7}{1.062^{12} - 1} = 70.210,91$$

c) $P = P_c$ is the IV of a 12-year annuity-due with constant installment $R_p = 55,000$:

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19 A strong limitation for the meaning of this calculus, and all those concerning perpetuities with constant installment, follows the unrealistic hypothesis of periodic constant profits in an unlimited time. Furthermore, if we suppose profit changing every p years in geometric progression, i.e. varying with constant rate, then – as we will see in the case of annuity with varying rates – Fisher's equation permits exact calculation by means of constant annuities, which are equivalent to those varying in geometric progression, if one assumes a new evaluation rate as a function of the given one and of the one in progression.

$$P_c = \frac{R_{12}\, i_{12}}{1+i_{12}} = \frac{55000 \cdot 1.062^{12}}{1.062^{12}-1} = 106{,}973.51$$

4) A field with poplars is bought by a private person just after the harvest and following reforestation. He leases the field to a forestry company for 4 productive cycles; if the cut is every 8 years, the length is 32 years. The rent on the basis of market prices is €43,500 to pay after each cut, with the tenant bearing the cost of reforestation. Calculate the IV of such a contract at the annual evaluation rate of 4.5%. In addition, in the hypothesis that the owner is able to obtain from the tenant the same total amount, but divided into annual advance installments, calculate the percentage increments of the contract value.

A. The IV of the standard contract is that of an 8-year temporary annuity-immediate for 32 years; thus, according to (5.17'),

$$V_0^{(1/8)} = R_8\, a_{\overline{32}|0.045}^{(1/8)} = 43{,}500\, \frac{1-1.045^{-32}}{1.045^8-1} = 77{,}858.22$$

The IV $\hat{V}_0^{(1/8)}$ of the contract is that of an annual annuity-due, temporary for 32 years, with installment $R_8/8$. Therefore, using $d = 0.0430622$, we obtain

$$\frac{R_8}{8}\, \ddot{a}_{\overline{32}|0.045}^{(1/8)} = 5{,}437.50\, \frac{1-1.045^{-32}}{0.0430622} = 5{,}437.50 \cdot 17.5443913 = 95{,}397.63$$

and thus the percentage increment is

$$100\, \frac{\hat{V}_0^{(1/8)} - V_0^{(1/8)}}{V_0^{(1/8)}} = 22.527\ \%$$

5.2.4. *Inequalities between annuity values with different frequency: correction factors*

We have seen that, in relation to all the unitary annuities considered so far, by changing frequency their IV are in inverse relation to the corresponding per period interest or discount intensities. Therefore, denoting by $j(1/p)$ and $\rho(1/p)$ the p-annual interest and discount intensities and by m the frequency of the fractional annuity, given using the compound regime we obtain: $\rho(1/p) < d < \rho(m) < \delta < j(m) < i < j(1/p)$, the following inequalities hold:

$$\ddot{a}_{\overline{n}|i}^{(1/p)} > \ddot{a}_{\overline{n}|i} > \ddot{a}_{\overline{n}|i}^{(m)} > a_{\overline{n}|i}^{(\infty)} > a_{\overline{n}|i}^{(m)} > a_{\overline{n}|i} > a_{\overline{n}|i}^{(1/p)}; (n \leq \infty) \qquad (5.19)$$

Analogous inequalities hold for the PVDA with same deferment r and for the FV of temporary unitary annuities.

Formulae shown in section 5.2.3 enable direct calculation of the capital value of non-annual annuities. Furthermore, it can be convenient to use *correction factors* to apply to the IV or to the installments of annual annuities, if these elements are easily available, to obtain, by multiplying, the IV or the FV or the *equivalent* installments of fractional or pluriannual annuities.

The correction factor to go from the IV to the FV of a temporary annuity for n years of whichever type is, in all cases, $(1+i)^n$ and it is the reciprocal for the inverse transformation.

More complex are the factors to go from annual annuities to fractional or pluriannual ones, and vice versa.

In light of this, we distinguish between two problems on such transformations, from frequency 1 (*annual* case) to frequency m with $m\text{-}1 \in \mathcal{N}$ (*fractional* case) or $m=1/p$ (*pluriannual* case with payments every p years).

PROBLEM A.– *Transformation of the capital value (at a given time) of the annual annuity in that of the annuity with the same length and annual rate but with frequency m (or, more generally, with a frequency changing from m' to m'') which leaves unchanged the total annual payment*[20].

PROBLEM B.– *Transformation of the installment of the annual annuity in that of the annuity with the same length and annual rate but with frequency m (or, more generally, with a frequency changing from m' to m'') which leaves unchanged the capital value (at a given time).*

Problem A is solved by applying to the capital value the correction factor f_A given by the reciprocal of the ratio between the corresponding per period intensities, i.e.:

– for annuity-immediate, $f_\mathrm{A} = i/j(m) > 1$ (being $i = j(1)$), applying it to the value of the annual annuity, and $f_\mathrm{A} = j(m')/j(m'')$ in general;

– for annuity-due, $f_\mathrm{A} = d/\rho(m) < 1$ (being $d = \rho(1)$), applying it to the value of the annual annuity[21], and $f_\mathrm{A} = \rho(m')/\rho(m'')$ in general.

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20 Observe that corresponding annuities in the sense of Problem A have installments proportional to the periods.

21 In practice this factor is seldom used, it is preferred to apply the factor $(1+i)^{1/m}\, i/j_{(m)}$ to the value of the annual annuity-immediate.

Problem B is solved by applying to the capital value the correction factor f_B given by the ratio between the corresponding per period rates, i.e.:

– for annuity-immediate, $f_B = i_{1/m}/i < 1/m$, applying it to the annual rate R, and $f_B = i_{1/m}''/i_{1/m}'$ in general;

– for annuity-due, $f_B = d_{1/m}/d > 1/m$, applying it to the annual rate $\ddot{R}$, and $f_B = d_{1/m}''/d_{1/m}'$ in general.

The installments for different frequencies, obtained solving Problem B, can be said to be *equivalent* because they are obtained by proportionality at different rates. The argument still holds if a regime different from the compound regime is used.

Reciprocal factors are applied for inverse transformations.

The factors for Problem A are directly justified, on the basis of the expressions for the values of the annuities considered in sections 5.2.2 and 5.2.3, observing that inverse proportionality exists between such values and the per period delayed or advance intensities with corresponding fractioning. Limiting ourselves to the IV of a temporary annuity (given that in all other cases the development is the same, changing only the numerator of the ratios), by indicating with R the annual total of the payments that remains unchanged, we have:

– for annuity-immediate: $R\,a_{\overline{n}|i}^{(m)} = R\,a_{\overline{n}|i}\ i/j(m)$; $R\,a_{\overline{n}|i}^{(m'')} = R\,a_{\overline{n}|i}^{(m')}\ j(m')/j(m'')$;

– for annuity-due: $R\,\ddot{a}_{\overline{n}|i}^{(m)} = R\,\ddot{a}_{\overline{n}|i}\ d/\rho(m) = R\,a_{\overline{n}|i}^{(m)}\ (1+i)^{1/m} = R\,a_{\overline{n}|i}$
$(i/j(m))(1+i)^{1/m} = R\,a_{\overline{n}|i}\ i/\rho(m)$ (i.e. $i/\rho(m)$ is correction factor from $a_{\overline{n}|i}$ to $\ddot{a}_{\overline{n}|i}^{(m)}$);
$R\,\ddot{a}_{\overline{n}|i}^{(m'')} = R\,\ddot{a}_{\overline{n}|i}^{(m')}\ \rho(m')/\rho(m'')$

In the case of *pluriannual annuities*, it is obvious that the correction factor to go from the IV of the annual annuity-immediate to the p-annual one, if -immediate is $i/j(1/p)$, if -due is $i/\rho(1/p)$.

As concerns the factors from problem B, it is obvious that $R_{1/m}$ is the installment of an annuity m-fractional -immediate equivalent (in the sense of the equality of capital values) to an annual annuity-immediate with installment R if and only if it is the installment of accumulation in one year of the amount R. Thus, the result is:
$R\,\sigma_{\overline{m}|i_{1/m}}\,1/m = R\dfrac{i_{1/m}}{(1+i_{1/m})^m} = R\dfrac{i_{1/m}}{i}$ [22]. We obtain an analogous result in general.

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22 Note that, $i_{1/m}/i$ being the installment to be paid at the end of each mth year to capitalize at the end of the year the unitary capital, it is also the correction factor to be applied to the

With annuity-due, $\ddot{R}_{1/m}$ is the installment of a m-fractional annuity-due equivalent to the annual annuity-due with installment $\ddot{R}$ if and only if it is the advance installment of amortization in one year of the amount $\ddot{R}$. Therefore we have:

$$\ddot{R}_{1/m} = \ddot{R}\,\ddot{\alpha}_{\overline{m}|i_{1/m}} = R\,\frac{d_{1/m}(1+i_{1/m})^m}{(1+i_{1/m})^m-1} = R\,\frac{d_{1/m}}{i}(1+i) = R\,\frac{d_{1/m}}{d}$$

We obtain an analogous result in general.

It is obvious that if we transform the annual delayed installment (see footnote 20), $\ddot{R}_{1/m}$ is the corresponding accumulation advance installment, thus the correction factor is $\dfrac{i_{1/m}}{i(1+i_{1/m})}$.

In the case of *pluriannual annuity*, the correction factor to go from the annual delayed installment to the p-annual one, if -immediate, is $s_{\overline{p}|i}$ and, if -due, is $a_{\overline{p}|i}$[23].

EXAMPLE 5.2.– Using the data in exercise 5.3, as $i = 0.0938069$ and thus $d = 0.085768$, $\rho_{12} = 0.08933$, the unitary annual and monthly annuity are

$$a_{\overline{10}|i} = 6.3116048;\ a^{(12)}_{\overline{10}|i} = 6.5784744;\ \ddot{a}^{(12)}_{\overline{10}|i} = 6.6278135;\ \ddot{a}_{\overline{10}|i} = 6.9035675$$

and thus the transformations using the correction factors are easily verified:

$$a^{(12)}_{\overline{10}|i} = a_{\overline{10}|i}\,0.0938069/0.09 \quad;\quad \ddot{a}^{(12)}_{\overline{10}|i} = \ddot{a}_{\overline{10}|i}\,0.08933/0.0857618$$

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delayed annual installment of amortization or accumulation in a prefixed number of years to obtain the equivalent delayed m-fractional installment of amortization or accumulation in the same number of years. We obtain an analogous result for advance payments, considering the fractional installment $d_{1/m}/d$.

23 If the annual installment is in advance, it is enough to use respectively $\ddot{s}_{\overline{p}|i}$ or $\ddot{a}_{\overline{p}|i}$. This can be verified directly using $m=1/p$ or simply observing that the annual installment is an installment of accumulation in p years of the amount given by the p-annual delayed installment or else installment of amortization in p years of the amount given by the p-annual advance installment.

The unitary continuous annuity is $a^{(\infty)}_{10|i} = 6.6032175$ and the inequalities are verified as

$$a_{10|i} < a^{(12)}_{10|i} < a^{(\infty)}_{10|i} < \ddot{a}^{(12)}_{10|i} < \ddot{a}_{10|i}$$

Exercise 5.8

1) Calculate the IV of the 10-year 4-fractional unitary annuity, both -immediate and -due, at the annual rate of 5%, knowing that the annual annuity-immediate value is 7.1217349.

A. By applying the correction factors with given rates and times, we obtain

$$a^{(12)}_{10|0.05} = a_{10|0.05} \frac{0.05}{0.0490889} = 7.8650458$$

$$\ddot{a}^{(12)}_{10|0.05} = \ddot{a}_{10|0.05} \frac{0.05 \, (1.05)^{1/4}}{0.0490889} = 7.9615675$$

2) Solve Problem A with data from exercise 5.3, maintaining the annual total of €7,800 and calculating the IV of the quarterly annuity-immediate through the correction factor on the IV of the monthly one (see Example 5.1).

A. We have $i = 0.0938069$ and thus $j(4) = 0.0906767$; the following is obtained:

$$a^{(4)}_{10|i} = a^{(12)}_{10|i} \frac{j(12)}{j(4)} = 6.5784744 \; \frac{0.09}{0.0906767} = 6.5293807$$

3) Solve Problem A of question 2 but referring to annuity-due.

A. We have $\rho(4) = 0.0886667$ and thus $\rho(12) = 0.08933$; the following is obtained:

$$\ddot{a}^{(4)}_{10|i} = \ddot{a}^{(12)}_{10|i} \frac{\rho(12)}{\rho(4)} = 6.627813 \; \frac{0.0893300}{0.0886667} = 6.6773945$$

4) Solve Problem B with data from exercise 5.3, calculating the monthly installment equivalent to the annual one of €7,800, both in the -immediate and -due cases.

A. With delayed installments, using $R = 7,800$, the result is

$$R_{1/12} = R \frac{i_{1/12}}{i} = 7,800 \; \frac{0.0075}{0.0938069} = 623.62.$$

With advance installments, given $\ddot{R} = 7{,}800$ and being $d_{1/12} = 0.0074442$ from which $d = 0.0857618$, we obtain

$$\ddot{R}_{1/12} = \ddot{R}\,\frac{d_{1/12}}{d} = 7{,}800\,\frac{0.0074442}{0.0857618} = 677.04$$

5) We have to amortize (in the sense specified in section 5.1) the debt $V_0 = €50{,}000$ over 5 years at the annual rate of 8.75% with delayed annual installments R. Calculate the constant installments. To amortize with six-monthly delayed or advance installments, calculate the value using correction factors.

A. The annual installment is $R = V_0\ \alpha_{\overline{5}|0.0875} = 12{,}771.35$, and thus the six-monthly equivalent delayed and advance installments are:

$$R_{1/2} = R\frac{i_{1/2}}{i} = 12771.35 \cdot 0.4895164 = 6{,}251.79$$

$$\ddot{R}_{1/2} = R\,\frac{i_{1/2}}{i(1+i_{1/2})} = 6251.79 \cdot 0.9589266 = 5{,}995.00$$

6) We have to accumulate (in the sense specified in section 5.1) a capital sum of $€37{,}500$ in 8 years at the annual rate of 6.15% with constant installments, annual delayed or quarterly. Calculate these installments.

A. The annual installment is $R = V_8\ \sigma_{\overline{8}|0.0615} = 3{,}768.50$ and then the quarterly equivalent delayed or advance installments are

$$R_{1/4} = R\frac{i_{1/4}}{i} = 3768.50\ 0.244329 = €921.15$$

$$\ddot{R}_{1/4} = R\,\frac{i_{1/4}}{i(1+i_{1/4})} = 3768.50 \cdot 0.2408129 = €907.50$$

Exercise 5.9

1) We have to amortize, at the annual rate of 6.60%, a debt of 1,450,000 monetary units (MU) with constant annual delayed installments over 20 years. To evaluate the convenience of an amortization with four-yearly installments, solve Problem B calculating the equivalent delayed and advance installment.

A. The annual installment is $R = V_0\ \alpha_{\overline{20}|0.066} = 1450000 \cdot 0.0914786 = 132643.95$. The 4-yearly equivalent delayed and advance installments are

$$R_4 = R \ \sigma_{\overline{4}|0.066} = 132643.95 \cdot 4.4137115 = 585{,}452.13$$

$$\ddot{R}_4 = R \ a_{\overline{4}|0.066} = 132643.95 \cdot 3.4180241 = 453{,}380.22$$

2) Solve Problem B of question 1) in circumstances where the annual constant amortization installment is advance.

A. The annual installment is $\ddot{R} = V_0 \ddot{\alpha}_{\overline{20}|0.066} = 1450000 \cdot 0.0858148 = 124{,}431.48$. The 4-annual equivalent delayed and advance installments are

$$R_4 = \ddot{R} \ \ddot{\sigma}_{\overline{4}|0.066} = 124431.48 \cdot 4.7050165 = 585{,}452.16$$

$$\ddot{R}_4 = \ddot{R} \ \ddot{\alpha}_{\overline{4}|0.066} = 124431.48 \cdot 3.6436137 = 453{,}380.25$$

Obviously the values R_4 and $\ddot{R}_4$ are the same in the results of 1) and 2) (except for rounding-off errors), due to the decomposability of the financial law used.

5.3. Evaluation of constant installment annuities according to linear laws

5.3.1. *The direct problem*

We have already mentioned that uniform financial laws, different from the compound laws, are sometimes used to evaluate annuities. It is worth studying the problem in detail.

As seen in Chapter 2, the need for simplicity leads us to use, for short lengths of time, the simple interest law in an accumulation process and the simple discount law in a discounting process[24]. Thus for some applications the following questions are relevant:

– the initial evaluation of an annuity on the basis of the simple discount law;

– the final evaluation of an annuity on the basis of the simple interest law.

Although the reader is referred to section 5.5 for the case of general installments, we give here the most important formulae for the case of *m-fractional annuities* (owing to short times) *with constant installments* and, always fixing 0 as the beginning of the annuity's interval, let us give the following definitions:

– $m > 1$ = annual frequency of payments;

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24 For them the exchange factor is linear, and they are called *linear laws*. Their conjugate, with hyperbolic factors, are usually used for indirect problems, e.g. offsetting, etc.

– s = total number of payments;

– i = interest intensity = annual interest rate;

– d = discount intensity = annual discount rate;

– R = delayed constant installment;

– $\ddot{R}$ = advance constant installment.

If the payments are *delayed*, the hth amount R is paid at time h/m, and the IV on the basis of the simple discount (SD) law at the rate d and the FV in s on the basis of the simple delayed interest (SDI) at rate i are given, respectively, by

$$V_0 = s\,R\,\left(1 - d(s+1)/2m\right)\ ;\ V_s = s\,R\,\left(1 + i(s-1)/2m\right) \tag{5.20}$$

Instead, if payments are in *advance*, the hth amount $\ddot{R}$ is paid at time $(h-1)/2m$, and the IV on the basis of the SD law at rate d and the FV in s on the basis of the SDI law at rate i are given, respectively, by

$$\ddot{V}_0 = s\,\ddot{R}\,\left(1 - d(s-1)/2m\right)\ ;\ \ddot{V}_s = s\,\ddot{R}\,\left(1 + i(s+1)/2m\right) \tag{5.21}$$

Equations (5.20) and (5.21) are obtained from the sum of terms in arithmetic progression. More simply, observing that $t' = (s-1)/2m$ is the average length of accumulation of the s delayed installments and the average length of the discounting of the s advance installments, while $t'' = (s+1)/2m$ is the average length of accumulation of the s advance installments and the average length of the discounting of the s delayed installments, we obtain:

$$V_0 = s\,R\,(1 - d\,t'')\ \ ;\ V_s = s\,R\,(1 + i\,t') \tag{5.20'}$$

$$V_0 = s\,\ddot{R}\,(1 - d\,t')\ ;\ V_s = s\,\ddot{R}\,(1 + i\,t'') \tag{5.21'}$$

equivalent to (5.20) and (5.21).

Exercise 5.10

1) We have to build up a fund of €12,000 with 10 constant delayed or advance monthly payments in a saving account at 6% annual in the SDI regime: calculate the value of each installment.

A. In the case of delayed payments, from the 2nd expression of (5.20) the following is obtained

$$R = 12,000 / \left[10 \ (1 + 0.06 \frac{9}{24}) \right] = \text{€}1,173\,59$$

The difference of 264.10 between the accumulated amount of €12,000 and the total payments, which amount to €11,735.90, is due to the interest accrued in the fund. In the case of advance payments, due to the 2nd expression of (5.21) the installment is

$$\ddot{R} = 12,000 / \left[10 \ (1 + 0.06 \frac{11}{24}) \right] = \text{€}1,167.88$$

2) In a hire purchase the client accepts 10 quarterly delayed payments of €400 each. If the seller is able to discount the payments at the annual rate of 8% in a simple discount regime, calculate the amount obtainable by the seller.

A. The obtainable amount is equal to the initial value V_0 given by the 1st expression of (5.20). This results in

$$V_0 = 4,000 \left(1 - 0.08 \frac{11}{8} \right) = 3,560$$

The spread of €440 with respect to thetotal payments of €4,000 is the amount of discount, as reward for the advance availability.

5.3.2. *Use of correction factors*

If the SDI law is used with factor $u(t) = 1 + it$ and the m-fractional annuity is considered with accumulation of interest only at the end of the year, the correction factors to be applied to the annual delayed installment R to have the equivalent m-fractional delayed installment $R_{1/m}$ or advance installment $\ddot{R}_{1/m}$, are obviously

$$delayed \ case: \ f_p = \frac{1}{m + \dfrac{m-1}{2} i} \ ; \ advance \ case: \ f_a = \frac{1}{m + \dfrac{m+1}{2} i} \qquad (5.22)$$

The correction factor is also the periodic installment for the accumulation of unit capital in one year. In fact, considering the temporal interval between 0 and 1, at

time 1 the FV V_1 of the annual payment is R, while the FV $V_1^{(m)}$ of the delayed m-fractional payments is $R_{1/m}m$ (for the principal) + $R_{1/m}\dfrac{m(m-1)}{2}i$ (for the interest); therefore, under the constraint we obtain $R_{1/m}/R = f_p$ given by the first formula in (5.22). With advance m-fractional payments we have $\ddot{V}_1^{(m)} = \ddot{R}_{1/m}m(1+\dfrac{m+1}{2}i)$ and the constraint $V_1 = V_1^{(m)}$ implies $\ddot{R}_{1/m}/R = f_a$ given by the second formula in (5.22).

5.3.3. *Inverse problem*

Equations (5.20) and (5.21) have been presented for the solution of the *direct problem*, consisting of the evaluation of the *initial value* and *final value* of an annuity given according to linear law. However, the same formulae solve univocally the *inverse problem*, consisting of:

– the calculation of the constant delayed (or advance) installment of amortization of the debt V_0 (or $\ddot{V}_0$) with a simple discount law;

– the calculation of the constant delayed (or advance) installment of capital funding V_s (or $\ddot{V}_s$) with a simple interest law.

Amortization and accumulation are usually carried out with such laws for short durations.

Exercise 5.11

We have to extinguish a debt of €5,000 at 9% annually in 3 years with delayed annual installments in the annual compound regime. There is the choice to amortize the debt with constant monthly delayed or advance installments with the assumption that the payments during the year produce *simple* interest, which only at the end of the year are accumulated and used for the amortization. Calculate the installments.

A. The amount for the annual installment is $R = 5,000\ \alpha_{\overline{3}|0.09} = 1,975.27$.

Using the correction factors (5.22) on R, the following values for the other are obtained:

– monthly delayed: $R_{1/12} = R\,f_p = 1,975.27/(12+\dfrac{11}{2}0.09) = 1,975.27 \cdot 0.08003 =$ 158.08;

– monthly advance: $\ddot{R}_{1/12} = \ddot{R} \, f_a = 1{,}975.27 / (12 + \dfrac{13}{2} 0.09) = 1{,}975.27 \cdot 0.07946$
$= 156.95$

In the monthly compound regime it would be:

$$i_{1/m} / i = 0.0800814; \text{ monthly delayed installment} = 158.18$$

$$i_{1/m} (1 + i)^{-1/m} / i = 0.0795083; \text{ monthly advance installment} = 157.05$$

5.4. Evaluation of varying installments annuities in the compound regime

5.4.1. *General case*

For the evaluation in the compound regime of annuities with varying installments, with the same signs, we can follow the classification shown in section 5.2. Here we will deal with such an argument, showing the similarities, but taking into account that the schemes based on the regularity of installments are not conserved.

Thus, we will limit ourselves to developing the calculus for the IV V_0 of annual temporary annuities-immediate (for n years), as for the other annuity schemes it is enough to take into account that, starting from the previous case, we can apply the following changes, valid with varying installments as well as constant installments:

– in the *fractional* (or *pluriannual*) case it is enough to use in the formulae, instead of years and annual rate, the number of payments and the equivalent per period rate;

– in the *-due* case each amount is paid one year before, thus the IV is $V_0(1+i)$;

– in the *delayed*[25] case each amount is paid after r years, thus the PVDA are $V_0(1+i)^{-r}$;

– the FV V_n is given by $V_0(1+i)^n$;

– in the *perpetuities* case it is enough to use $n \to +\infty$, but such a calculation can be carried out only if a rule on the formation of installments in an unlimited time is given.

By putting together the previous five rules we can obtain all the results of the classification seen in section 5.2 if the aforementioned value V_0 has been calculated.

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25 It is almost unnecessary to observe that, in the case of varying installments, the annuity-immediate value coincides with that of the corresponding annuity-due delayed by one period.

Denoting by R_h or $\ddot{R}_h$ (in chronological order) the different delayed or advance installments the annuity is the union of the concordant dated amounts $\bigcup_{h=1}^{n}(h, R_h)$ or $\bigcup_{h=1}^{n}(h-1, \ddot{R}_h)$. Considering from now on the case of non-negative installments (and at least one positive one), due to (4.3") the IV of the annuity-immediate or -due are:

$$V_0 = \sum_{h=1}^{n} R_h \, (1+i)^{-h} \quad ; \quad \ddot{V}_0 = \sum_{h=1}^{n} \ddot{R}_h \, (1+i)^{-(h-1)} \tag{5.23}$$

Proceeding analogously, the FV of the annuity-immediate or -due are

$$V_n = \sum_{h=1}^{n} R_h (1+i)^{n-h} \quad ; \quad \ddot{V}_n = \sum_{h=1}^{n} \ddot{R}_h (1+i)^{n-h+1} \tag{5.24}$$

EXAMPLE 5.3.– Applying (5.23) and (5.24), calculate the IV and FV of annuities-immediate or -due. An Excel spreadsheet can be used and the installments put directly into columns, applying recurrent formulae, such as

– $V_{h-1} = (R_h + V_h)(1+i)^{-1}$ from $V_n = 0$, to calculate pro-reserves and IV in the -immediate case;

– $V_{h-1} = \ddot{R}_{h-1} + V_h(1+i)^{-1}$ from $V_n = 0$, to calculate pro-reserves and IV in the -due case;

– $M_h = M_{h-1}(1+i) + R_h$ from $M_0 = 0$ to calculate retro-reserves and FV in the -immediate case;

– $M_h = (M_{h-1} + \ddot{R}_{h-1})(1+i)$ from $M_0 = 0$, to calculate retro-reserves and FV in the -due case.

The following table is obtained where, both in the -immediate and the -due case, the IV is given by the pro-reserve in 0 and the values below in the column give the pro-reserve for the following years, while the FV is given by the retro-reserve in n as credit for the counterpart which pays the installments, and the values above in the column give the retro-reserve for the preceding years. Obviously, given that an annuity operation is unfair, the retro-reserves and pro-reserves will never coincide in the various years.

	Rate = 0.042			*Length =* 10		
Year	delayed installment	advance installment	delayed pro-reserve	advance pro-reserve	delayed retro-reserve	advance retro-reserve
0	0.00	521.44	5,230.48	5,450.16	0.00	0.00
1	521.44	412.36	4,928.72	5,135.73	521.44	543.34
2	412.36	125.61	4,723.37	4,921.75	955.70	995.84
3	125.61	1,544.98	4,796.14	4,997.58	1,121.45	1,168.55
4	1,544.98	897.33	3,452.60	3,597.61	2,713.53	2,827.50
5	897.33	69.55	2,700.28	2,813.69	3,724.83	3,881.27
6	69.55	587.11	2,744.14	2,859.39	3,950.82	4,116.76
7	587.11	897.54	2,272.28	2,367.72	4,703.87	4,901.43
8	897.54	1,258.32	1,470.18	1,531.93	5,798.97	6,042.53
9	1,258.32	285.10	273.61	285.10	7,300.85	7,607.48
10	285.10	0.00	0.00	0.00	7,892.58	8,224.07
	IV annuity-immediate =	5,230.48				
	IV annuity-due =			5,450.16		
	FV annuity immediate =				7,892.58	
	FV annuity due. =					8,224.07

Table 5.1. *Pro-reserves and retro-reserves in the immediate and due case*

The Excel instructions are as follows. C1: 0.042; F1: 10; use the first two rows for data and column titles, the annual values from 0 to 10 are in rows 3-13:

– column A (year): A3: 0; A4:= A3+1; copy A4, then paste on A5-A13;

– column B (installments in the -immediate case): B3: 0; from B4 to B13: (insert data: delayed installments);

– column C (installments in the -due case): copy from B4 to B13, then paste on C3 to C12 (insert data: advance installments); C13: 0;

– column D (pro-reserve in the -immediate case): D13: 0; D12: = (B13+D13)*(1+C$1)^-1; copy D12, then paste backwards on D11 to D3;

– column E (pro-reserve in the -due case): E13: 0; E12: = C12+E13*(1+C$1)^-1; copy E12, then paste backwards on E11 to E3;

– column F (opposite of the retro-reserve in the -immediate case): F3: 0; F4: = F3*(1+C$1)+B4; copy F4, then paste on F5 to F13;

– column G (opposite of the retro-reserve in the -due case): G3: 0; G4: = (G3+C3)*(1+C$1); copy G4, then paste on G5 to G13; (initial and final value of annuities-immediate and -due): D15: = D3; E16: = E3; F17: = F13; G18: = G13.

Continuous flow

In case of continuous flow $\varphi(t)$ of annuity from 0 to n, the IV and the FV are expressed respectively by

$$\overline{V}_0 = \int_0^n \varphi(t)e^{-\delta t}dt \tag{5.23'}$$

$$\overline{V}_n = \int_0^n \varphi(t)e^{\delta(n-t)}dt \tag{5.24'}$$

With the previous formulae the *direct problem* is solved by finding the IV or the FV of an annuity with varying installments. The same formulae form a constraint for the *inverse problem*, by finding an annuity, i.e. a sequence of dated amounts with the same sign, which has a given IV or FV. Thus, as already seen regarding annuities with constant installments, if the IV is given, we have a *problem of gradual amortization* of an initial debt, while if the FV is given, we have a *problem of gradual funding* at the end of the time interval. In the case of constant installments we obtain a unique solution, owing to n-1 equality constraints between the installments. Instead, in general the solution of the inverse problem is not unique, having n-1 degrees of freedom. Furthermore, in the amortization, due to technical and juridical reasons, inequality constraints are introduced so that the amortization installments cover at least the accrued interests.

5.4.2. *Specific cases: annual annuities in arithmetic progression*

Let us here consider some relevant models, which refer to specific cases of annual annuities with varying installments. Among them, we can consider the installment evolution in arithmetic progression (AP). We obtain such a feature when all the subsequent installments vary according to a constant rate γ (positive or negative) of the first installment R. Thus, the subsequent differences are constant, and are given by the *ratio D*. Therefore, $D = \gamma R$ and

$$R_h = R + (h\text{-}1)D > 0, (h = 1, \ldots, n) \tag{5.25}$$

Let us focus first on the *normalized* or *unitary* annuity, also called an *increasing annuity*, where the first installment and the ratio are unitary; therefore $R_h = h$. In the *temporary case* the IV is indicated with the symbol $(Ia)_{\overline{n}|i}$; its value is

$$(Ia)_{\overline{n}|i} = \sum_{h=1}^{n} h\,(1+i)^{-h} = \frac{1}{i\,d}\left[1-(1+n\,d)(1+i)^{-n}\right] \tag{5.26}$$

For *perpetuities* ($n = \infty$), prompt and delayed, given that $\lim\limits_{n\to+\infty} n(1+i)^{-n} = 0$, we obtain the IV of an *increasing perpetuity*

$$(Ia)_{\overline{\infty}|i} = \sum_{h=1}^{+\infty} h\,(1+i)^{-h} = \frac{1}{i\,d} \;\; ; \;\; {}_{r/}(Ia)_{\overline{\infty}|i} = \frac{v^r}{i\,d} \tag{5.26'}$$

Denoting by $(Is)_{\overline{n}|i} = (1+i)^n\,(Ia)_{\overline{n}|i}$ the FV in the delayed case, in the other cases the symbols for the values of the *increasing annuities* are easily extended as in section 5.2.

To deduce the closed form given in (5.26) and then in (5.26'), some algebraic developments are needed. However, it is also possible to use financial equivalences, which we will use starting from perpetuities. Let us observe, first, that the relation $a_{\overline{n}|i} = a_{\overline{\infty}|i} - {}_{n/}a_{\overline{\infty}|i}$ (see footnote 16) can be generalized as follows

$$(Ia)_{\overline{n}|i} = (Ia)_{\overline{\infty}|i} - {}_{n/}(Ia)_{\overline{\infty}|i} - n\,{}_{n/}a_{\overline{\infty}|i} \tag{5.27}$$

(because $(Ia)_{\overline{\infty}|i} - {}_{n/}(Ia)_{\overline{\infty}|i}$ is the IV of a perpetuity, increasing until time n but with constant installments after n; thus to obtain the IV of a temporary annuity we still need to subtract $n\,{}_{n/}a_{\overline{\infty}|i}$). We have an analogous conclusion for the -due case.

The 1st part of (5.26') is justified for the transitivity property of the equalities. We can observe, in fact, that using the delayed evaluation rate i (equivalent to the advance rate $d = i/(1+i)$), the supply $(0,S)$ is equivalent to the perpetuity of its advance interests, i.e. $\bigcup_{h=0}^{+\infty}(h,Sd)$ with graph

$$
\begin{array}{ccccccc}
0 & & Sd & & Sd & & Sd \\
\rule{0pt}{0pt} & \text{--}\,\text{--} & & \text{--}\,\text{--} & & \text{--}...\text{--}\text{--} & \quad\text{--}\,\text{--} \\
0 & & 1 & & 2 & & h
\end{array}
$$

Furthermore, each supply (h,Sd) is equivalent to the perpetuity, delayed by h years, of its delayed interests Sid, i.e. $\bigcup_{k=h+1}^{+\infty}(k,Sid)$ with graph

$$
\begin{array}{ccccc}
Sid & & Sid & & Sid \\
\rule{0pt}{0pt} & \text{--}\,\text{--} & & \text{--}...\text{--}\text{--} & \quad\text{--}\,\text{--} \\
h+1 & & h+2 & & k
\end{array}
$$

and adding to h, for all supplies, the triangular development $\bigcup_{h=0}^{+\infty} \bigcup_{k=h+1}^{+\infty} (k, Sid)$
$= \bigcup_{k=1}^{+\infty} (k, kSid)$ is obtained, with graph

$$[\begin{matrix} 0 \\ 0 \end{matrix} -- \begin{matrix} Sid \\ 1 \end{matrix} -- \begin{matrix} 2Sid \\ 2 \end{matrix} --...-- \begin{matrix} kSid \\ k \end{matrix} --].$$

This last annuity, for which the IV is $Sid\ (Ia)_{\overline{\infty}|i}$ [26], is equivalent to $(0,S)$ for which the IV is S. Therefore, $Sid\ (Ia)_{\overline{\infty}|i} = S$, i.e.: $(Ia)_{\overline{\infty}|i} = 1/id$. This proves the 1st part of (5.26'). The 2nd part is obvious. Developing (5.27), we obtain

$$(Ia)_{\overline{\infty}|i} - {}_n/(Ia)_{\overline{\infty}|i} - n\,{}_n/a_{\overline{\infty}|i} =$$

$$= \frac{1}{i\,d} - \frac{(1+i)^{-n}}{i\,d} - n\frac{1-(1+i)^{-n}}{i} = \frac{1}{i\,d}\left[1-(1+n\,d)(1+i)^{-n}\right]$$

i.e. (5.26).

After what has been said about the relationship between the different cases, this is an exercise to give the expressions for the other values of the annual *increasing annuity*. Starting from (5.26) and (5.26'), it is found that

$$(Is)_{\overline{n}|i} = \frac{(1+i)^n}{i\,d}\left[1-(1+n\,d)(1+i)^{-n}\right] \quad (I\ddot{s})_{\overline{n}|i} =$$

$$\frac{(1+i)^n}{d^2}\left[1-(1+n\,d)(1+i)^{-n}\right];$$

---

[26] Considering that, by definition, $(Ia)_{\overline{\infty}|i}$ is the IV of $\bigcup_{k=1}^{+\infty}(k,k)$, by multiplying the amounts by Sid the IV of the annuity $\bigcup_{k=1}^{+\infty}(k,kSid)$ is obtained. We ascertain here the strength of the compound discount: at whichever rate, the present value of an annuity with diverging installment and infinite length is finite, i.e. it is in no case diverging! This is due to the fact that an increasing exponential function becomes infinite faster than a linear one, and also a polynomial one. Therefore, the result also holds true for *increasing* perpetuities of the higher order, which we can define for subsequent sums, in this manner: the hth installment of the perpetuity-due with IV $1/d^m$ ($m>2$) is the sum of the first h installments of the annuity for which the IV is $1/d^{m-1}$. With $m=3$, $1/d^3$ is the IV of the perpetuity with advance installments which are the partial sums of the installments' sequence corresponding to the IV $1/d^2$, i.e.: 1; 1+2=3; 1+2+3=6, ..., $1+...+n = n(n+1)/2$, ...

$$(I\ddot{a})_{\overline{n}|i} = \frac{1}{d^2}\left[1-(1+n\,d)(1+i)^{-n}\right] \;;$$

$$(I\ddot{a})_{\overline{\infty}|i} = \frac{1}{d^2}\;;$$

$$_{r/}(I\ddot{a})_{\overline{n}|i} = \frac{v^r}{d^2}\left[1-(1+n\,d)(1+i)^{-n}\right]$$

$$_{r/}(I\ddot{a})_{\overline{\infty}|i} = \frac{v^r}{d^2}$$

EXAMPLE 5.4.– To have an order of magnitude, let us give in Table 5.2 the IV, PVDA, FV for two parametric scenarios.

Type of annuity	Symbol	$(i=4.20\%;\,r=5)$		$(i=11.35\%;\,r=2)$	
		$n=20$	$n=\infty$	$n=9$	$n=\infty$
-immediate IV	$(Ia)_{\overline{n}|i}$	122.141386	590.702948	23.457999	86.436774
-due IV	$(I\ddot{a})_{\overline{n}|i}$	127.271304	615.512472	26.120482	96.247340
-immediate PVDA	$_{r/}(Ia)_{\overline{n}|i}$	99.431559	480.87316	18.919538	69.713696
-due PVDA	$_{r/}(I\ddot{a})_{\overline{n}|i}$	103.607684	50.069840	21.066905	77.626194
-immediate FV	$(Is)_{\overline{n}|i}$	278.110396	∞	61.730949	∞
-due FV	$(I\ddot{s})_{\overline{n}|i}$	289.791032	∞	68.737412	∞

Table 5.2. *IV, PVDA, FV calculation*

In addition, for the two scenario perpetuities with *n* and *i*, we obtain

$_{n/}(Ia)_{\overline{\infty}|i}$	259.426752	32.846309
$_{n/}(I\ddot{a})_{\overline{\infty}|i}$	270.322676	36.574365
$_{n/}a_{\overline{\infty}|i}$	10.456740	3.348052
$_{n/}\ddot{a}_{\overline{\infty}|i}$	10.895924	3.728056

(5.27) is verified with these parameters in the two scenarios, distinguishing between annuity-immediate and annuity-due:

1st scenario, annuity-immediate: 590.702948 – 259.426752 – 20·10.456740 = 122.141386
1st scenario, annuity-due: 615.512472 – 270.322676 – 20·10.895924 = 127.271324
2nd scenario, annuity-immediate: 86.436774 – 32.846309 – 9·3.348052 = 23.457999
2nd scenario, annuity-due: 96.574365 – 36.574365 – 9·3.72805 = 26.120482

We can now give the formulae for an annual annuity in AP which is characterized by the couple (R,D). These formulae generalize those of the *increasing annuity.* For the reasons already mentioned, we can consider only the IV and FV of an annuity-immediate, temporary for n years. The following is obtained:

$$V_0 = R\ a_{\overline{n}|i} + D\ _{1/}(Ia)_{n-\overline{1}|i} = (R\text{-}D)\ a_{\overline{n}|i} + D\,(Ia)_{\overline{n}|i} \qquad (5.28)$$

$$V_n = V_0\,(1+i)^n = (R\text{-}D)\ s_{\overline{n}|i}\ + D\,(Is)_{\overline{n}|i} \qquad (5.28')$$

In fact, by definition, particularizing (5.23) with R_h given by (5.25):

$$V_0\ =\ \sum_{h=1}^{n} R_h\ (1+i)^{-h}\ =\sum_{h=1}^{n}\big[R+(h\text{-}1)D\big]\,(1+i)^{-h} = R\,a_{\overline{n}|i} + D\,_{1/}(Ia)_{n-\overline{1}|i}$$

The last side of (5.28) follows from the simple identity: $R+(h\text{-}1)D = (R\text{-}D)+ hD$.

Exercise 5.12

A lease of a company has been agreed between the parties for an annual rent, delayed for 12 years, of €285,000 for the first year, withan annual increment of 3% of the initial rent. At the compound annual rate of 6.20%, calculate the IV of such an annuity.

A. By applying (5.28) where: $i=0.062$; $d=0.058380$; $v=0.941620$; $R=285000$; $D=8,550$, the following is the result

$$V_0 = R\,a_{1\overline{2}|0.062}\ + D\,(1.062)^{\text{-}1}\,(Ia)_{1\overline{1}|0.062} =$$

$$= 285{,}000{\cdot}8.292677 + 8{,}550{\cdot}0.941620{\cdot}33.856815 = 2{,}635{,}989.12$$

5.4.3. *Specific cases: fractional and pluriannual annuities in arithmetic progression*

The linear variability of the installments of an annuity in AP is in practice more frequently applied using fractional installment.

Let us observe, first, that the fractioning can concern both the *frequency of payments k, and the frequency of variations h*, where $h \in \mathcal{N}$, $k=wh$ being $w \in \mathcal{N}$ the number of consecutive unchanged payments. Considering that formulae to

generalize (5.28) are needed to obtain the IV of annuities in AP[27], it is sufficient here to extend to the *fractional* case the *increasing annuity* that will enter into the calculations, through an appropriate *normalization* that is convenient to apply *so as to maintain the increments in the annual total of payments as unitary.*

Given the above, it is easy to verify that:

– the *increasing* fractional annuity-immediate (-due) with *annual* increment of the installment (proportionally to a natural number), i.e. with $h=1$, $k=w>1$, is formed by installments payable at the end (beginning) of each kth of year and of amount $1/k$ in the 1st year, $2/k$ in the 2nd year, etc. Generalizing (5.26), the IV of annuities-immediate is

$$(Ia)_{\overline{n}|i}^{k|1} = \frac{1}{j(k)\,d}\left[1-(1+n\,d)(1+i)^{-n}\right]; \quad (Ia)_{\overline{\infty}|i}^{k|1} = \frac{1}{j(k)\,d} \qquad (5.29)$$

which is obtainable by applying to the value of the annual annuity the same correction factor $i/j(k)$ (= ratio between intensities) already used for constant annuity. The same factor has to be applied also for FV and PVDA;

– the *increasing* fractional annuity-immediate (-due) with $h>1$, $k=wh>1$, is formed, due to the aforementioned normalization, by installments payable at the end (beginning) of each kth of year, so that, for the -immediate case, the first w payments of the 1st year are of amount $1/hk$, the second w payments of the 1st year are of amount $2/hk$,... the last w payments of the 1st year are of amount $1/k$,.... the first w payments of the nth year are of amount $[(n-1)h+1]/hk$, the second w payments of the nth year are of amount $[(n-1)h+2]/hk$, …. the last w payments of the nth year (in the case of a temporary annuity for n years) are of amount n/k. The IV of this annuity-immediate is

$$(Ia)_{\overline{n}|i}^{k|h} = \frac{1}{j(k)\rho(h)}\left[1-(1+n\,\rho(h))(1+i)^{-n}\right]; (Ia)_{\overline{\infty}|i}^{k|h} = \frac{1}{j(k)\rho(h)} \qquad (5.30)$$

and generalizes (5.26) in the sense that the annual intensities i and d are substituted in (5.30) for those relative to frequency k and h. The same thing holds true for FV and PVDA;

– if all the payments of an *increasing* fractional annuity-immediate, with $h=1$ or $h>1$, are backdated for $1/k$ of a year, we obtain an *increasing* fractional annuity-due, the IV of which follows from that of the annuity-immediate on multiplying by

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27 In generalizing (5.28) for the fractional case it is convenient to consider its last term, at least when $h>1$, which implies installment variations during the year, to avoid the complication of deferment for a fraction of years.

$(1+i_k)$; therefore it is sufficient to substitute $j(k)$ with $\rho(k)$ into the formulae in (5.29) and (5.30).

Proof

Let us first observe that (5.27) is generalized in

$$(Ia)_{\overline{n}|i}^{k|h} = (Ia)_{\overline{\infty}|i}^{k|h} - {}_{n/}(Ia)_{\overline{\infty}|i}^{k|h} - n\,{}_{n/}a_{\overline{\infty}|i}^{(k)} \qquad (5.27')$$

Therefore, to prove (5.29) and (5.30) it is sufficient to consider the perpetuities, because using their value we obtain that of the temporary annuities. (5.29) is proved observing that, analogously to what was seen for the annual annuity, an amount *S* in 0 is equivalent to the annual perpetuity, starting with 0, of advance interest *Sd* and that each installment *Sd* is equivalent to the subsequent *k*-fractional perpetuity of delayed interest $Sdi_{1/k}$. The total of the payments is, therefore:

– $S\,d\,i_{1/k}$ at the end of each period with duration $1/k$ of the 1st year;

– $2\,S\,d\,i_{1/k}$ at the end of each period with duration $1/k$ of the 1st year;

– etc.

Therefore it is sufficient to use $S=1/j(k)d$ in order to obtain (5.29) as IV of the annuity with fractional payments $1/k$ in the 1st year, $2/k$ in the 2nd year, etc., and thus unitary increments in the annual total of payments, which is 1 in the 1st year.

Equation (5.30), which generalizes (5.29), is proved observing that the supply $(0,S)$ is equivalent to the subsequent *h*-fractional perpetuity-due with constant installments $Sd_{1/h}$, each of which is equivalent to the following *k*-fractional annuity-immediate of constant installments $Sd_{1/h}i_{1/k}$. The amount *S* is thus the IV of the perpetuity with payments:

– $Sd_{1/h}i_{1/k}$ at the end of each of the first k/h periods with duration $1/k$ of the 1st year;

– $2Sd_{1/h}i_{1/k}$ at the end of each of the second k/h periods with duration $1/k$ of the 1st year;

– $hSd_{1/h}i_{1/k}$ at the end of each of the last k/h periods with duration $1/k$ of the 1st year;

– $(h+1)Sd_{1/h}i_{1/k}$ at the end of each of the first k/h periods with duration $1/k$ of the 2nd year;

– $(h+2)Sd_{1/h}i_{1/k}$ at the end of each of the second k/h periods with duration $1/k$ of the 2nd year;

$-2hSd_{1/h}i_{1/k}$ at the end of each of the last k/h periods with duration $1/k$ of the 2nd year;

– etc.

It is sufficient to use $S = 1/[j(k)\rho(h)]$ in order to obtain the perpetuity that starts with the fractional payment $1/(hk)$, reaching the level n/k after n years, for which the IV is given by the 2nd of (5.30).

It is obvious that the IV of the annuity, temporary (or perpetuity), which has installments proportional to those of an *increasing* fractional annuity-immediate (with $h>1$, $k=wh>1$) and first payment H, is obtained from the first (or second) value in (5.30) multiplying by Hhk. More generally, the IV of whichever fractional annuity in AP is obtained with an appropriate linear combination of the unitary IV $a_{\overline{n}|i}^{(k)}$ and $(Ia)_{\overline{n}|i}^{k|h}$.

Observation

When $h>1$, the value of payments of the 1st year is no longer unitary; its value is $T_1 = (h+1)/2h$. In general the total payment of the year $s+1$ is

$$T_{s+1} = \frac{k}{h}\left(\frac{sh+1}{hk} + ... + \frac{(s+1)h}{hk}\right) = \frac{h+1}{2h} + s, \ s = 0,1,2,... \tag{5.31}$$

thus $T_{s+1} = T_s+1$, $\forall(s,h)$ and the unitary normalization of the annual increments is verified. The total of payments in the first n years is

$$T^{(n)} = \sum_{s=0}^{n-1}T_{s+1} = \frac{n}{2}\left(n+\frac{1}{h}\right) \tag{5.31'}$$

Continuous increasing annuity

The values in the continuous case are obtained, as usual, on diverging the frequency. However, in this case we have two frequencies: the frequency of payments and the frequency of increments.

Recalling that $\delta = \lim_{k\to\infty} j(k) = \lim_{h\to\infty} \rho(h)$, we observe that there is no distinction between -due and -immediate in the case of varying installments as well. Let us give the results, that can be easily proved, starting from (5.30), in both cases:

– if only the payment frequency k diverges, the IV is

$$(Ia)_{\overline{n}|i}^{\infty|h} = \frac{1}{\delta\ \rho(h)}\left[1-(1+n\ \rho(h))(1+i)^{-n}\right]; \ (Ia)_{\overline{\infty}|i}^{\infty|h} = \frac{1}{\delta\ \rho(h)} \tag{5.32}$$

– if both the payment frequency k and the increment frequency h diverge, the IV is

$$(Ia)_{\overline{n}|i}^{\infty|\infty} = \frac{1}{\delta^2}\left[1-(1+n\ \delta)(1+i)^{-n}\right] \ ; \ (Ia)_{\infty|i}^{\infty|\infty} = \int_0^{+\infty} t\ e^{-\delta\ t}dt = \frac{1}{\delta^2} \qquad (5.32')$$

and the total of payments in the first n years is $n^2/2$.

EXAMPLE 5.5.– Let us make some numerical comparisons on the fractional *increasing* annuity, changing the fractioning with the constraint $h \le k$, verifying the increasing behavior if using the same h from higher deferment to higher anticipation, decreasing if h increases, fixing the other parameters. Let us assume i=0.07; n=10. Considering the frequency 1, 4, 12, the equivalent values are: d=0.0654206, $j(4)$=0.0682341, $j(12)$=0.0678497, $\rho(4)$=0.0670897, $\rho(12)$=0.0674683, δ=0.0676586 and the following table is obtained, where $T^{(10)}$ is the maximum value obtainable (h,k) at zero rate.

k	h	$T^{(10)}$	$(Ia)_{\infty|i}^{k|h}$	$(I\ddot{a})_{\infty|i}^{k|h}$
1	1	55.000	34.7390688	37.1707813
4	1	55.000	35.6381167	36.2460231
12	1	55.000	35.8400231	36.0426277
∞	1	55.000	35.9412524	35.9412524
4	4	51.250	32.8980119	33.4591782
12	4	51.250	33.0843943	33.2714212
∞	4	51.250	33.1778403	33.1778403
12	12	50.417	32.4783090	32.6619097
∞	12	50.417	32.5700432	32.5700432
∞	∞	50.000	32.2671080	32.2671080

Table 5.3. *Comparisons on the fractional increasing annuities*

Exercise 5.13

1) An industrial company has to pay a monthly delayed rent for leasing (equipment, etc., see section 6.5) equal to the amortization installment at the interest rate of 9.50% for 7 years proportional to the initial value of the plant of €48,500, net of 5% of the value initially paid as an advance, and without any other clause except for an annual increment of 12% on the initial rent. Calculate the rents for the 7 years and the initial value of the borrowed amount at the evaluation rate for the supposed income of 12% per year.

A.

a) *calculation of the installments*: Borrowed amount = M = 48,500 (1–0.05) = €46,075; for (5.13), being j_{12} = 0.0910984, initial rent = C = $M/(12\,a^{(12)}_{\overline{7}|0.095})$ = 46,075/(12·5.1615977) = €743.87.

Given the adjustment clause there is an annual increment of the monthly rent of €18.60 resulting in:

– Monthly rent in the 1st year = €743.87

– Monthly rent in the 2nd year = €762.47

– Monthly rent in the 3rd year = €781.07

– Monthly rent in the 4th year = €799.67

– Monthly rent in the 5th year = €818.27

– Monthly rent in the 6th year = €836.87

– Monthly rent in the 7th year = €855.47

A.

b) *calculation of the* IV: using (5.13), (5.29) and: n=7; i=0.12; k=12; h=1; R = 743.874·12 = 8926.488; D = 18.597·2 = 223.164 (in terms of annual flows) and generalizing (5.28) in

$$V_0 = R\,a^{(k)}_{\overline{n}|i} + D\ _1/(Ia)^{k|h}_{\overline{n}|i} = (R\text{-}D)\,a^{(k)}_{\overline{n}|i} + D\,(Ia)^{k|h}_{\overline{n}|i},$$

the following is obtained:

$$V_0 = 8926.488·4.8096288 + 223.164·13.7441973 = €46,000.30$$

2) A three-year work contract starting on the 1st January has an annual wage bill of €13,390 to be paid in 12 delayed monthly salaries + 13th salary for Christmas, and also an increasing benefit to add to the 12 monthly, initially equal to 5% of the initial salary but with quarterly increments all equal to it, not affecting the 13th one. To calculate the end of job indemnity, let us value the FV of such a contract at an evaluation rate of 6.60%.

A. It is convenient to first calculate the IV, afterwards accumulating it for 3 years, and keep separate the 13th salary from the ordinary monthly salaries, including the increasing benefit.

The payments for the 13th salary give rise to an annual annuity-immediate with 3 installments of 13,390/13 = €1,030, with IV of: $V'_0 = 1030\, a_{\overline{3}|0.066} = €2,722.92$.

The ordinary monthly salaries give rise to a 12-fractional *increasing* temporary annuity with quarterly increments, formed by 3 delayed monthly installments of €1,081.50 followed by another 3 of €1,133.00, etc. The parameters are $n=3$; $h=4$; $k=12$; $i=0.066$. Thus applying the last term of (5.28), its IV is

$$V''_0 = 1,030\cdot 12\cdot a^{(12)}_{\overline{3}|0.066} + 1,030\cdot 0.05\cdot 4\cdot 12\cdot (Ia)^{12|4}_{\overline{3}|0.066} = 31,784.29 + 10,027.48 =$$

$$= €41,811.77$$

Thus, the IV of the contract is, at the rate of 6.6%, $V_0 = V'_0 + V''_0 = €44,534.69$ and the FV is $V_3 = 1066^3\, V_0 = €53,947.34$.

Pluriannual increasing annuities

Let us briefly mention the pluriannual *increasing* annuities, from which can be deduced with linear combination the IV of the pluriannual annuities in AP, which find practical application generalizing the annuities with constant installments. Let us consider only the case of $h=k=1/p$ (i.e. p-annual *increasing* annuity with period $1/k$ and increment after each payment)[28]. The following normalization implies that the sth installment is sp^2; thus the rth, at the end of n years, if $n=rp$, is $rp^2=np$. Therefore, the IV are obtained using (5.30), where the expressions already used in (5.18) for the *p*-annual intensities $j(1/p) = [(1+i)^p-1]/p$; $\rho(1/p) = [1-(1+i)^{-p}]/p$ are taken into account. For the perpetuities we obtain the following results

$$(Ia)^{\frac{1}{p}|\frac{1}{p}}_{\infty|i} = \frac{1}{j(1/p)\rho(1/p)} = \frac{p^2}{(1+i)^p + (1+i)^{-p} - 2} \tag{5.30'}$$

$$(I\ddot{a})^{\frac{1}{p}|\frac{1}{p}}_{\infty|i} = \frac{1}{\rho^2(1/p)} = \frac{p^2}{\left[1-(1+i)^{-p}\right]^2}$$

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[28] The conclusions for this specific case can be easily obtained from those of the annual *increasing* annuity (see (5.26) and (5.26')) assuming as the new unit measure the p-year and thus adopting proper measures for time and rates.

while for the temporary case it is enough to multiply (5.30') by [1-$(1+n\rho(1/p))(1+i)^n$]. To have the values of the annuity, proportional to the previous one, for which the first installment is H, it is sufficient to multiply by H/p^2.

Exercise 5.14

Consider again the problem of Exercise 5.7 assuming that the cost of the plants due to the five-yearly replacements increases by 26% in respect to the initial cost.

A. The parameters are: $n=20$; $p=5$; $i=7.45\%$ thus $\rho_{(1/5)} = (1\text{-}1.0745^{-5})/5 = 0.0603638$. The cost for the plant at time 0 is €255,000; the increment of cost for each replacement is: $255,000 \cdot 0.26 = €66,300$ starting at time 5 for 3 times. To avoid deferments let us divide the five-yearly varying cost by the sum of a five-yearly advance cost of $255,000\text{-}66,300 = €188,700$ and by an increasing cost proportional to an *increasing* five-yearly annuity-due with a first installment of €66,300. Thus, applying (5.17) and (5.30') modified for the temporary case, the IV of such an operation is

$$V_0 = \frac{188,700}{5} n^{(1/5)}_{20|0.0745} + \frac{66,300}{25} (In)^{\frac{1}{5}|\frac{1}{5}}_{20|0.0745} = 37,740 \cdot 5 \cdot \frac{1-1.0745^{-20}}{1-1.0745^{-5}} +$$

$$+ \frac{66,300}{25} \frac{25}{\left[1-1.0745^{-5}\right]^2} [1\text{-}(1+20 \cdot 0.0603638)(1.0745)^{-20}] = 37,740 \cdot 12.6298497 +$$

$$+ 2,652 \cdot 130.5014439 = 476,650.53 + 346,089.83 = €822,740.36$$

5.4.4. *Specific cases: annual annuity in geometric progression*

Temporary annuities

Often in annuities the installment variation is proportional to a fixed ratio of the previous installment instead of the initial one. It follows that the behavior of installments is in geometric progression (GP), the ratio of which we will indicate with q. Typical are those phenomenon of adjustment with constant rate: if a salary increases at the rate of 5% the following index numbers are obtained

100, 105, 110.75, 115.7625, 121.5506, etc.

in GP with ratio $q=1.05$.

Let us define a *temporary annual unitary annuity-immediate in geometric progression* with ratio $q>0$ with the following operation:

$$\bigcup_{h=1}^{n}(h,q^{h-1}) \tag{5.33}$$

with graph: $\begin{bmatrix} 1 & q & & q^{n-1} \\ - & - & - - \dots - & - \\ 1 & 2 & & n \end{bmatrix}$.

If the period is not annual, it can always be assumed to be a unit measure of time (and in such case i is the corresponding per period rate), thus unifying the treatment.

The IV of the unitary annuity (5.33) in a compound regime is given by

$$(Ga)_{\overline{n}|i}^{[q]} = \sum_{h=1}^{n} q^{h-1} v^{h} = \begin{cases} nv \ , & \text{if } q = 1+i \\ v\dfrac{1-(qv)^{n}}{1-qv}, & \text{if } q \neq 1+i \end{cases} \tag{5.34}$$

If we have an unitary annuity-due, its IV $(G\ddot{a})_{\overline{n}|i}^{[q]}$ is obtained multiplying the values in (5.34) by $(1+i)$ and

$$(G\ddot{a})_{\overline{n}|i}^{[q]} = \sum_{h=1}^{n} q^{h-1} v^{h-1} = \begin{cases} n \ , & \text{if } q = 1+i \\ \dfrac{1-(qv)^{n}}{1-qv}, & \text{if } q \neq 1+i \end{cases} \tag{5.34'}$$

More generally, the IV of annuities in GP, -immediate or -due, with a first installment equal to R are given by

$$V_0 = R(Ga)_{\overline{n}|i}^{[q]}; \qquad \ddot{V}_0 = R(G\ddot{a})_{\overline{n}|i}^{[q]} \tag{5.34''}$$

Using $\eta = q\text{-}1$ (=algebraic rate of variation of the GP), if $q<1+i$ i.e. $\eta<i$, qv is the discount factor at the rate $\lambda = (1/qv)\text{-}1 > 0$ such that the IV (5.34') is also that of a constant annuity-due at the rate λ (see (5.2)). If instead $q>1+i$ i.e. $\eta>i$, qv is the accumulation factor at the rate $\mu = qv\text{-}1 > 0$ such that the IV (5.34') is also the FV of a unitary constant annuity-immediate at the rate μ (see (5.6)). In formulae[29]:

29 See the observation in footnote 19. We obtain a formula analogous to the 1st expression of (5.35) for annuity-immediate if it is normalized assuming the first installment equal to q, coherently with the following viewpoint: considering the annuity in g.p. $\bigcup_{k=0}^{n}(k,q^{k})$, the IV of the annuity-due is calculated on the first n supplies; the IV of the annuity-immediate takes account of the following n supplies after the first one. For all choices of q and i, the two

$$q<1+i: \ (Gn)^{[q]}_{\overline{n}|i} = \ddot{a}_{\overline{n}|\lambda} \ ; \ q>1+i: \ (G\ddot{a})^{[q]}_{n|i} = s_{\overline{n}|\mu} \tag{5.35}$$

where

$$q = 1+\eta = \frac{1+i}{1+\lambda} = (1+i)(1+\mu) \tag{5.36}$$

Equation (5.35) speeds up the calculation of (5.34) and (5.34') leading it back to that of the values of constant annuities.

Due to the decomposability, the FV (Gs), $(G\ddot{s})$ and the *p.v.d.a* . $_{r/}(Ga), _{r/}(G\ddot{a})$ of unitary annuities in GP, -immediate and -due, are obtained from IV with the usual factors:

$$(Gs)^{[q]}_{n|i} = (1+i)^n (Ga)^{[q]}_{n|i} \ ; \ (G\ddot{s})^{[q]}_{n|i} = (1+i)^n (G\ddot{a})^{[q]}_{n|i} \tag{5.37}$$

$$_{r/}(Ga)^{[q]}_{n|\ i} = (1+i)^{-r}(Ga)^{[q]}_{n|\ i} \ ; \ _{r/}(G\ddot{a})^{[q]}_{n|\ i} = (1+i)^{-r}(G\ddot{a})^{[q]}_{n|\ i} \tag{5.38}$$

From a general point of view, let us consider annuities with installments that are sum of two addends: the former is constant, the latter is varying in GP Considering an annual temporary annuity-immediate (or with another period to assume as unitary) with installment $R_h = H + Kq^{h-1}$, its IV and FV are

$$\begin{cases} V_0 = H \, a_{\overline{n}|i} + K(Ga)^{[q]}_{n|\ i} \\ V_n = H \, s_{\overline{n}|i} + K(Gs)^{[q]}_{n|\ i} \end{cases} \tag{5.39}$$

Analogous formulae hold for other types of annuities in GP.

Real and monetary variations

The formulation that leads to (5.36) is a specific case – which considers rates that are constant in time – of the problem of financial evaluation with rates that vary in time and with variation of the purchasing power of money. Such a problem, which has an important application in macroeconomics and finance, can be shown with a simple argument. Let m_t and c_t be the interest rate for the year $(t-1,t)$, on the

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theoretical rates λ and μ introduced in (5.35), because of (5.36) are linked by the relation $(1+\lambda)(1+\mu) = 1$, thus they have opposite signs. Also, we have to consider the case $q=1+i$, in which $\lambda = \mu = 0$.

monetary market and on the commodity market (such as wheat, for example) respectively, in the sense that:

– for M euros loaned in t-1, we pay back $M(1+m_t)$ euros in t;

– for C kilograms of wheat loaned in t-1, we give back $C(1+c_t)$ kilograms of wheat in t.

In addition, let r_t be the variation rate of the wheat price in euros, i.e. C kilograms are traded today for M euros and after one year for $M(1+r_t)$ euros. It is obvious that the three rates m_t, c_t, r_t are bound by an equation, which is deduced as follows. If at time t-1 the C kilograms of wheat are traded on the market for M euros, two equivalent loans of C and M lead in t to the equivalent return of $C(1+c_t)$ kilos and $M(1+m_t)$ euros; but in such time C kilos are traded with $M(1+r_t)$ euros, and thus $C(1+c_t)$ kilograms are traded with $M(1+r_t)(1+c_t)$ euros. For comparison the multiplicative relation is found, which is also called *Fisher's equation*[30],

$$1 + m_t = (1 + r_t)(1 + c_t) \tag{5.40}$$

Supposing a market economy with only one commodity (wheat), m_t is the monetary interest rate (or *rate in value*), c_t is the real interest rate (or *rate in volume*), r_t is the variation rate of the commodity price. (5.40) thus expresses the market constraint in terms of exchange annual factors. If r_t and c_t are small, the product $r_t c_t$ in the development of $(1+r_t)(1+c_t)$ can be ignored and (5.40) can be approximated using the simple relation

$$m_t = r_t + c_t \tag{5.40'}$$

usually used (and sometimes abused) in the description of macroeconomic phenomena.

In the specific case of constant rates, putting $m=i$, $c=\lambda$, $r=\eta$, (5.40) is reduced to (5.36) and adding the effects for n years, the 1st expression of (5.35) is found, which expresses the equality between: a) the IV at rate i (which acts as the monetary rate m) of the annuity in GP with ratio q, i.e. with variation rate η, and: b) the IV at rate λ, which act as real rate c, of the constant annuity[31]. This also holds in cases in

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30 See Fisher (1907).
31 If there is a devaluation of the commodity compared to the money, then $0<q<1$, while in the case of appreciation it is $q>1$. If and only if the real rate λ is positive, then $\eta<i$, while $\lambda=0$ implies $\eta=i$. It can happen that the rate of price increment is higher than the monetary interest rate, so we obtain a real rate $\lambda<0$. In this last case, to avoid the use of a negative rate in the formulae, it is enough to introduce the value μ linked to λ by (5.36) and to apply (5.35).

which all the installments are multiplied by the constant R. Therefore, we have to consider equivalent to a discount:

– at a real rate λ, a constant annuity with installments given by the constant values of fixed quantities at the constant price of the initial year; or

– at the corresponding monetary rate i, the annuity of the varying values of the same quantities at the current prices that vary at the rate η[32].

Perpetuities

If the annuity in GP is a perpetuity – differently from what happens for perpetuity linearly increasing[33] or according to powers with integer exponent greater than 1 – of time, its IV assumes a finite value only if $q<1+i$. We obtain in such a case: $\lim\limits_{n\to\infty}[q^n(1+i)^{-n}]=0$ and thus

$$(Ga)^{[q]}_{\infty|i} = \frac{v}{1-qv} \;;\; (G\ddot{a})^{[q]}_{\infty|i} = \frac{1}{1-qv} \tag{5.41}$$

In general, with an initial installment R_1, the IV V_0 is obtained from (5.41) and multiplying by R_1. The other values are obtained simply by applying the corresponding factors.

Exercise 5.15

1) Let a loan have paid back delayed installments indexed at 3%, the first of which coincides with the constant amortization installment of the debt of €140,000 over 10 years at the rate of 6.3%. Calculate the sequence of installments and the IV of the temporary annuity and the IV of the corresponding perpetuity, if it is finite.

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32 To generalize the 1$^{\text{st}}$ expression of (5.35) in the case of installments R_h and varying rates m_t, r_t, c_t it is enough to replicate Fisher's equation (5.40) using $t=1,2,..,n$, and we obtain with simple developments the equality

$$\sum_{h=1}^{n} R_h \prod_{t=1}^{h} \left(\frac{1+r_t}{1+m_t}\right)^h = \sum_{h=1}^{n} R_h \prod_{t=1}^{h} \left(\frac{1}{1+c_t}\right)^h$$

between the IV at the monetary rates m_t of the varying amounts R_h indexed at the rate r_t, i.e. evaluating the commodity at the current prices, and the IV at the real rates c_t of the amounts R_h which are not indexed, i.e. evaluating the commodity at a constant price. If the rates are linked by (5.40'), the equality is an approximation.

33 We can observe that the arithmetic progression behavior is a discretization of the linear behavior while the geometric progression behavior is a discretization of the exponential behavior. It is important to analyze this comment thoroughly, from the viewpoint of the mathematical analysis, and the problems that come up when we consider perpetuities.

A. We obtain $qv = 0.9689558 < 1$, $R_1 = 140,000\,\alpha_{1\overline{0}|0.063}$, $R_h = 1.03\,R_{h-1}$ ($h=2,...,10$) and the following values in euros are found for the installments:

$R_1 = 19,292.79$; $R_2 = 19,871.57$; $R_3 = 20,467.72$; $R_4 = 21,081.75$; $R_5 = 21,714.20$

$R_6 = 22,365.63$; $R_7 = 23,036.60$; $R_8 = 23,727.70$; $R_9 = 24,439.53$; $R_{10} = 25,172.71$

Due to (5.34"), the IV is

$$V_0 = R_1(Ga)^{[1.03]}_{10|0.063} = 19,292.79\frac{1-\left(\frac{1.03}{1.063}\right)^{10}}{1.063\left(1-\frac{1.03}{1.063}\right)} = 19,292.79\cdot 8.1962415 = €158,128.37$$

which has to compare with the value 140,000 of the annuity with constant installment.

If we consider a perpetuity, using $q = 1.03 < 1+i = 1.063$, (5.41) is applied and the IV of the perpetuity is bounded and is

$$V_0 = R_1(Ga)^{[1.03]}_{\infty|0.063} = \frac{19,292.79}{1.063\left(1-\frac{1.03}{1.063}\right)} = 19,292.79\cdot 34.2414848 = €660,613.77$$

2) Assuming the compound regime and using the annual rate of 6%, let us consider a sequence of advance annual rent indexed at 9% for 10 years, the first of which coincides with the funding annual constant installment of the amount of €100,000 in 10 years. Calculate the IV and the FV of the aforementioned annuity, and also the rate of the equivalent constant annuity. Also consider the perpetuity.

A. The first advance installment is

$$R_1 = 100,000\ \ddot{\sigma}_{1\overline{0}|0.06} = 100,000\cdot 0.0715735 = 7,157.35$$

and the following installments are

$$R_2 = 1.09\,R_1 = 7,586.80;\ ....;\ R_{10} = 1.09^9\,R_1 = 12,092.20$$

By applying (5.34") and (5.37), the following is the result:

$$\ddot{V}_0 = R_1(G\ddot{a})^{[1.09]}_{10|6\%} = 7157.35\cdot 11.3746307 = €81,412.21$$

$$\ddot{V}_{10} = R_1(G\ddot{s})^{[1.09]}_{10|6\%} = 7157.35\cdot 20.3702312 = €145,796.87$$

The value €145,796.87, if compared with the capital of €100,000 accumulated with 10 constant installments of €7,157.35, shows the effect of the compound index at 9%.

The rate μ applied in (5.35) results here as (1.09/1.06)-1 = 0.0283019 independently from n and forming an active rate of accumulation. The constraint $(1+\lambda)(1+\mu) = 1$ holds true (see footnote 29), and thus if we exchange the two rates i and $\eta=q$-1, using the index at 6% and the interest at 9%, the following is obtained: $\lambda = (1.12/1.09)^{-1}$ -1 = 0.0283019, coinciding with μ but to be interpreted as the allowed amortization rate.

As $q>1+i$, not only the values of the temporary annuities, but also the single discounted values, increase with n, and thus the perpetuity has unlimited value.

5.4.5. *Specific cases: fractional and pluriannual annuity in geometric progression*

Proceeding analogously as for the annuities in AP, let us briefly examine the changes connected with the fractioning of annuities in GP.

This is useful because sequences of payments and variations subdivided during the year are widely used (see section 5.4.5). With the positions already used, let $h \in \mathcal{N}$ be the variation frequency and $k=wh$ the payment frequency, with $w \in \mathcal{N}$ being the number of consecutive unchanged payments.

To simplify the discussion without loss of generality, it is convenient to use $h=1$ and then $k=w$. This is obtained assuming as a *new unit measure of time, the period between two consecutive installment variations*, which we will call the *invariance period*, having fractioned the installment in k equal parts[34], with delayed and advance payments at each kth of the period; rate i will be the equivalent rate.

The normalization that leads to the *unitary fractional prompt* annuity is that in which, fractioning the payment of each period into k equal parts, the total of payments of the first period is unitary and those for the following periods proceeds in GP of ratio q, as shown in the following graph for an -immediate with temporary n

$$\left[\begin{array}{ccccccccc} \frac{1}{k} & \frac{1}{k} & & \frac{1}{k} & \frac{q}{k} & & \frac{q}{k} & \frac{q^2}{k} & & \frac{q^2}{k} & & \frac{q^{n-1}}{k} & & \frac{q^{n-1}}{k} \\ \frac{1}{k} & \frac{2}{k} & \cdots & \frac{1}{k} & \frac{k+1}{k} & \cdots & \frac{2k}{k} & \frac{2k+1}{k} & \cdots & \frac{k}{3k} & \cdots & \frac{(n-1)k+1}{k} & \cdots & \frac{nk}{k} \end{array}\right]$$

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34 With the symbols used in section 5.4.3, this is the case of $h=1$, $k=r$.

while for an -due the same amounts are backdated by $1/k$. Their IV are obtained from (5.34) and (5.41) by applying for each invariance period, and thus for the whole n years, the same correction factors f_A obtained in section 5.2.4 for *Problem A*, respectively $i/j(k)$ and $i/\rho(k)$[35].

With the symbols taking their obvious meanings, we obtain:

$$(Ga)^{[q;k]}_{\overline{n}|i} = \begin{cases} d\, n\, /j(k)\ , & \text{if } q = 1+i \\ \dfrac{d[1-(qv)^n]}{j(k)[1-qv]}, & \text{if } q \neq 1+i \end{cases} \tag{5.42}$$

$$(Ga)^{[q;k]}_{\infty|i} = \dfrac{d}{j(k)[1-qv]}, \quad \text{if } q < 1+i \tag{5.42'}$$

$$(G\ddot{a})^{[q;k]}_{\overline{n}|i} = \begin{cases} d\, n\, /\rho(k)\ , & \text{if } q = 1+i \\ \dfrac{d[1-(qv)^n]}{\rho(k)[1-qv]}, & \text{if } q \neq 1+i \end{cases} \tag{5.42''}$$

$$(G\ddot{a})^{[q;k]}_{\infty|i} = \dfrac{d}{\rho(k)[1-qv]}, \quad \text{if } q < 1+i \tag{5.42'''}$$

If all of the payments of the first invariance period are R, to obtain V_0, it is enough to multiply R by (5.42) or (5.42') in the immediate case, or else to multiply R by (5.42'') or (5.42'''). To obtain the FV and the PVDA of r periods, multiply by $(1+i)^n$ and v^r. For the IV in perpetuity, use $(qv)^n = 0$.

For Problem B the correction factors are those already used, i.e. those needed to obtain the k-fractional installments with an addendum in GP from one invariance period to the next. Therefore, in the -immediate case we can write the installments

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35 A proof based on equivalences is the following. For the $(h+1)$-th period we want the financial equivalence in $h+1$ (which is the instant of the end of the period and also the instant of payment of the per period non-fractional installment q^h) between such an installment and all of the fractional installments of the period, for which the valuation is

$$\Sigma^k_{r=1}\frac{1}{k}q^h(1+i_{1/k})^{k-r} = \frac{1}{k}q^h(1+i_{1/k})^k\Sigma^k_{r=1}(1+i_{1/k})^{-r} =$$

$$= \frac{1}{k}q^h(1+i_{1/k})^k(1+i_{1/k})^{-k}\frac{(1+i_{1/k})^k-1}{1+i_{1/k}-1} = q^h\frac{i}{j_k}$$

Therefore, also in this case: $f_A = \dfrac{i}{j_k}$.

as $R_h^{(k)} = H^{(k)} + K^{(k)} q^{h-1}$, so to not modify the values (5.39). In the -due case the results are analogous, obtaining $\ddot{H}^{(k)}$ and $\ddot{K}^{(k)}$ from $\ddot{H}$ and $\ddot{K}$. With the latter we have[36]

$$H^{(k)} = H \frac{i_{1/k}}{i}, K^{(k)} = K \frac{i_{1/k}}{i} \quad , \ddot{H}^{(k)} = \ddot{H} \frac{i_{1/k}}{i(1+i_{1/k})}, \ddot{K}^{(k)} = \ddot{K} \frac{i_{1/k}}{i(1+i_{1/k})} \quad (5.43)$$

For the *pluriannual case*, going back to the annual unit of measure, we only consider the case $h=k=1/p$, i.e. of a normalized p-annual annuity with variation at each payment; it is not restrictive to assume, to be consistent with the parameters of the annual annuity, the ratio q^p (that would be obtained by annually applying the ratio q). To calculate the IV of a temporary annuity it is sufficient to apply (5.34) and (5.34'), assuming as the unit of measure the interval of p years. Therefore, in terms of annual parameters, the following is easily obtained, in the -immediate case:

$$(Ga)_{n|i}^{[q^p;\frac{1}{p}]} = \begin{cases} n(1+i)^{-p}/p \ , & \text{if } q=1+i \\ \dfrac{1-(qv)^n}{(1+i)^p - q^p} \ , & \text{if } q \neq 1+i \end{cases} \qquad (5.44)$$

and in the -due case:

$$(G\ddot{a})_{n|i}^{[q^p;\frac{1}{p}]} = \begin{cases} n/p \ , & \text{if } q=1+i \\ \dfrac{1-(qv)^n}{1-(qv)^p} \ , & \text{if } q \neq 1+i \end{cases} \qquad (5.44')$$

The IV of a perpetuity assumes a finite value only if $q<1+i$, resulting, in such a case, in: $\lim_{n \to \infty} (qv)^n = 0$, and thus

$$(Ga)_{\infty|i}^{[q;\frac{1}{p}]} = \frac{1}{(1+i)^p - q} \quad ; \quad (G\ddot{a})_{\infty|i}^{[q;\frac{1}{p}]} = \frac{1}{1-(qv)^p} \qquad (5.45)$$

---

36 It is enough to observe that also here H, K , $\ddot{H}, \ddot{K}$ are the *FV* of the annuities in the invariance period with installments respectively $H^{(k)}, K^{(k)}, \ddot{H}^{(k)}, \ddot{K}^{(k)}$.

By using $q=1$, we go back to the values of constant annuities. The usual factors are used to obtain the FV and the PVDA In order to obtain the corresponding values of the effective annuities, multiply the values in (5.44), (5.44') or (5.45) for the first installment.

Continuous annuities in geometric progression

With continuous flows of payments, it is enough to consider two possibilities which, acting on the amplitude of the invariance periods, cover all cases:

a) *continuous constant flow* in each year (or more generally in each invariance period to which the parameters refer) and variations in GP with ratio q from one period to the next;

b) *varying continuous flow* in exponential way.

The normalized values for case a) are obtained from those of the k-fractional annuities with $k \to \infty$ (the non-normalized values are obtained by multiplying for the total of the first period). By applying the correction factor d/δ from (5.34) and (5.41) the IV are obtained.

$$(Ga)_{n|i}^{[q;\infty]} = \begin{cases} dn/\delta & \text{if } q = 1+i \\ \dfrac{d[1-(qv)^n]}{\delta(1-qv)} & \text{if } q \neq 1+i \end{cases} \; ; \; (Ga)_{\infty|i}^{[q;\infty]} = \frac{d}{\delta(1-qv)} \text{if } q < 1+i \text{ (5.46)}$$

To obtain the normalized values for case b), which give the highest continuity degree with $k \to \infty$, $h \to \infty$, let us first define the *variation intensity of continuous flow* (constant, because the payment flow evolves in an exponential way) given by $\psi = ln\, q$ which is consistent with the annual ratio $q = e^{\psi}$. Thus, the evolution of the discounted flows is given by $e^{(\psi-\delta)t}$ and the IV can be obtained using the integral calculus (analogously to the case of constant annuities: see footnote 17) obtaining[37]

37 We can write: $V_0 = \int_0^n e^{(\psi-\delta)t} dt = \dfrac{e^{(\psi-\delta)n}-1}{\psi-\delta} = \dfrac{(qv)^n-1}{\ln(qv)}$, if $q \neq 1+i$; $V_0 = n$ if $q = 1+i$,

that implies $\psi = \delta$. Recalling the Taylor series of $\ln(1+x)$, it is seen that $qv-1$ is the linear approximation of $ln(qv)$, which is very precise when $qv \cong 1$. In such cases, the difference between the normalized *IV* of annual annuity and of continuous annuity are negligible; in fact the change of deadlines does not bring practical effects because increasing the flow compensates the discount.

$$(G\overline{a})_{n|i}^{[q]} = \begin{cases} n\,, & \text{if } q=1+i \\ \dfrac{(qv)^n - 1}{\ln(qv)}\,, & \text{if } q \neq 1+i \end{cases} \quad ; \ (G\overline{a})_{\infty|i}^{[q]} = -1/\ln(qv),\ \text{if } q < 1+i \qquad (5.46')$$

The IV of a continuous perpetuity in GP in cases a) and b) assume a finite value only if $q<1+i$.

Exercise 5.16

Calculate the IV and FV at the 4-convertible annual rate = 0.056 on the three-yearly interval of validity of the contract, of the annuity given by a delayed monthly wage of a worker. This wage is set up by a fixed part of €1,700 and by a benefit initially at €400 and then increasing at the quarterly ratio of 0.8%. Compare the results with those of a continuous annuity with the same financial parameters.

A. It is convenient to assume the quarter period to be unitary and to use: fixed part = 5,100; initial benefit =1,200; ratio $q=1,008$; rate $i=0.056/4=0.014$; frequency of payments $k=3$; length $n=12$; thus: $v=0.9861933$; $d/j(3)=0.9907814$; $qv=0.9940828$; $(qv)^{12}=0.9312595$. Using (5.12) and (5.42) the IV is

$$V_0 = 5,100\, a_{12|0.014}^{(3)} + 1,200\, (Ga)_{12|0.014}^{[1.008;3]} = 5,100 \cdot 11.0267981 + 1,200 \cdot 11.5099724 =$$
$$56,236.67 + 13,811.97 = 70,048.64$$

The FV is

$$V_n = 1.014^{12}\, V_0 = 1.1815591 \cdot 70,048.64 = 82,766.61$$

For comparison, let us calculate, using the same parameters of amount and rate, the values in the case of continuous flow with continuous increments. Leaving the quarter as the unit measure of time and using (5.16) and (5.46'), the following is obtained

$$V_0 = 5,100\, \overline{a}_{12|0.014}^{(\infty)} + 1,200\, (G\overline{a})_{12|0.014}^{[1.008]} = 5,100 \cdot 11.0524121 + 1,200 \cdot 11.5826613$$
$$= 56,367.30 + 13,899.19 = 70,266.50$$

$$V_n = 1.014^{12}\, V_0 = 1.1815591 \cdot 70,266.50 = 83,024.02$$

The values of the constant continuous unitary annuity are only different by a small amount from those of the analogous monthly annuity and this also holds true for the varying annuity in GP given that $qv = 0.9940828 \cong 1$ (see footnote 37).

Exercise 5.17

An industrial company works at a plant for which the initial cost of €280,000 is already covered, but – expecting an average economic length of 5 years, without break-up value and with cost increments for periodic renewal at the annual compound rate of 5% – wants to cover the renewal cost in 20 years through semiannual delayed payment in a profitable bank fund at the compound rate of 6%. Calculate the semiannual payments:

a) using the hypothesis of constant payments in the 20 years;

b) using the hypothesis of payments increasing every 5 years in progression corresponding to the variation of the annual compound rate of 5%.

A. *Computation of cost of five-yearly renewals:*

– after 5 years: $C_1 = 280{,}000(1.05)^5 = €357{,}358.84$;

– after 10 years: $C_2 = C_1(1.05)^5 = €456{,}090.50$;

– after 15 years: $C_3 = C_2 (1.05)^5 = €582{,}099.89$;

– after 20 years: $C_4 = C_3 (1.05)^5 = €742{,}923.36$.

The outflows for such costs give rise to a pluriannual annuity-immediate in GP with $p = 5$; $n = 20$; $i = 0.06$; ratio $q^p = 1.05^5 = 1.2762816$, and the IV, due to (5.44), is

$$V_0 = 357{,}358.84 \ (Ga)_{20|0.06}^{[1.2762816,\frac{1}{5}]} = 357{,}358.84 \ \frac{1 - 0.990566^{20}}{1.06^5 - 1.05^5} =$$
$$= 357{,}358.84 \cdot 2.7878276 = €996{,}254.84$$

We now have to find the installments of the annuity to accumulate in the fund what is needed for the periodic renewal in the two cases a) and b) specified above.

Hypothesis a)

Assuming the year is the unit measure, using $i=0.06$; $n=20$; $m=2$ and using the correction factor $i_{1/2}/i$ specified in section 5.2.4, the semiannual delayed installment $R_{1/2}$ to deposit in a fund that provides the payments for the costs $C_1,...,C_4$ calculated above, is obtained. The following result holds true:

$$R_{1/2} = V_0\,\alpha_{\overline{n}|i}\ i_{1/2}/i = 996{,}254.84\cdot 0.0871846 \cdot 0.4927169 = €42{,}796.45$$

The values $R_{1/2}$ form constant outflows so as to balance over the 20 years the deposits in a fund with increasing costs; therefore during the 20 years a reserve is

formed, equal to the current balance and always to the credit of the depositing person and in debt of the institution that is managing the fund (so 6% is always a debit rate for this institution). This fund dies away after 20 years, as is confirmed by the following scheme of balances at the end of each five-years of the 20 years, where the FV of the five-yearly annuity at 6% annual of payments $R_{1/2}$ is 489,627.13. In the following table each row refers to a five-year period.

NG.	Existing balance (1)	FV of 5-year payments (2)	Updated balance (3) = (1)+(2)	Withdraw for renewal (4)	Residual of 5 - year period (5) = (3) – (4)	Fund after 5 years (6)=(5)1.06^5
1	0.00	489,627.13	489,627.13	357,358.84	132,268.30	177,004.82
2	177,004.82	489,627.13	666,631.96	456,090.50	210,541.46	281,751.97
3	281,751.97	489,627.13	771,379.11	582,099.89	189,279.22	253,298.29
4	253,298.29	489,627.13	742,925.43	742,923.36	(°) 2.07	

(°) Apparent final balance = 2.07 instead of 0, due to rounding-off.

Table 5.4. *Dynamics of a fund in hypothesis a)*

Hypothesis b)

In order to form the amount of five-yearly costs for renewal, we now have delayed constant semi-annual payments inside each five-year period, increasing when passing from one five-year period to the next with the same annual ratio of 5% with which the renewal costs increase. This implies, as we will verify, the balancing between the FV of the payments and the absorption of substitutions, already calculated, with consequent lack of residuals and thus zeroing of the reserve at the end of each five-year period. We can develop the calculation assuming the five-year period as a unit measure of time and solving in respect to K the first equation in (5.39), using: $H=0$; $i = (1.06)^5-1 = 0.3382256$; $n=4$; $V_0 = 996254.84$. This equation becomes

$$V_0 = K(Ga)_{20|0.06}^{[1.2762816;\frac{1}{5}]}$$

from which: $K=357,358.84$. This value is the equivalent FV of the semiannual payments $K^{(10)}$ of the first five-year period, which are obtainable by applying the correction factor $i_{1/10}/i=0.0973983$; therefore, we have $K^{(10)}=0.0973983\cdot357,358.84 = 31,235.38$. For the following five-year periods the semiannual payments and their five-year period FV increase in GP with ratio 1.2762816 every 5 years. The evolution of such payments in the four five-year periods and the verification of the

zeroing of residuals are shown in the following table, where each row is referred to a five-year period.

N.	Existing balance (1)	Semiannual payment (2)	FV of 5-year payments (3)	Withdrawal for renewal (4)	Residual of 5-year period (5)=(3) – (4)	Fund after 5 years (6)=(5)1,06
1	0	31,235.38	357,358.84	357,358.84	0	0
2	0	39,865.14	456,090.50	456,090.50	0	0
3	0	50,879.15	582,099.89	582,099.89	0	0
4	0	64,936.12	742,923.36	742,923.36	0	0

Table 5.5. *Dynamics of a fund in hypothesis b)*

Also in hypothesis b), the fund is never in credit because, due to the semiannual payments, it remains in debt inside each five-year period, but becomes 0 at its end and remains 0 during the first following half-year.

Exercise 5.18

1) Recall the second problem of exercise 5.15, which considers a temporary annual annuity-due in GP; using the same data let us consider the following variations:

a) annually varying continuous flow with the same progression;

b) continuous flow with continuous increments, given by $e^{\psi t}$, with $\psi = \ln 1.09$, discounted according to the intensity $\delta = \ln 1.06$.

A. *Case a)* The annual installment is substituted for the constant annual flow, equal to €7,157.35 during the 1st year, afterward in GP at 9% for 10 years, all evaluated in the compound regime with $i = 6\%$. Due to (5.46), the IV is obtained by applying the correction factor $d/\delta = 0.9714233$ to that of the annual case. We obtain: $V_0 = 7157.35 \cdot 11.0495810 = 79,085.72$. The FV is given by: $V_{10} = V_0(1.06)^{10} = 79085.72 \cdot 1.7908477 = 141,630.48$.

A. *Case b)* By applying (5.46'), $IV = V_0 = 7,157.35 \dfrac{(1.09/1.06)^{10} - 1}{\ln(1.09/1.06)} =$

$7,157.35 \cdot 11.5348438 = 82,558.91$ is obtained, and also FV $= V_{10} = V_0(1.06)^{10} = 82,558.91 \cdot 1.7908477 = 147,850.44$.

Compare the results obtained here with those from the second part of Exercise 5.15.

2) Calculate the IV for the normalized annuities as in part 2 of exercise 5.15 and part 1 of exercise 5.18, but assuming $q = 1.03$.

A. Using the formulae already discussed, the normalized IV are summarized in the following table.

Type of payment of the annuity	temporary (10 years)	perpetuity
Advance annual payments	8.8179309	35.3332994
Continuous flow with annual increments	8.5659496	34.3236102
Continuous flow with continuous increments	8.6925517	34.8309069

Table 5.6. *Calculation of the normalized IV*

5.5. Evaluation of varying installment annuities according to linear laws

5.5.1. *General case*

Also varying installments are often used for short periods linear exchange laws. Let us find here the IV and FV at time *s* of a *m-fractional annuity with varying installments*, considering again the symbols and assumptions of section 5.3, but indicating with R_h the hth delayed installment and with $\ddot{R}_h$ the hth advance installment. Such annuities are the operations $\hat{O}$ for which the supplies are (h/m, R_h) in the delayed case and ($(h-1)/m$, $\ddot{R}_h$) in the advance case.

Therefore, if the payments are *delayed*, the IV according to the SD law at rate *d* and the FV in *s* according to the SDI law at rate *i* are given, respectively, by

$$V_0 = \sum_{h=1}^{s} R_h\left(1 - d\,\frac{h}{m}\right); \ V_s = \sum_{h=1}^{s} R_h\left(1 + i\,\frac{s-h}{m}\right) \tag{5.47}$$

However, if the payments are in *advance*, the IV, according to the SD law at rate *d* and the FV in *s* according to the SDI law at rate *i*, are given, respectively, by

$$V_0 = \sum_{h=1}^{s} R_h\left(1 - d\,\frac{h}{m}\right) \ ; \quad V_s = \sum_{h=1}^{s} R_h\left(1 + i\,\frac{s-h}{m}\right) \tag{5.47'}$$

Equations (5.47) and (5.47') solve the already-mentioned *direct problem* (installment→value)[38], but we also have to consider here the *inverse problem* (value→installment) to solve amortization and accumulation calculus with varying installments according to linear laws. We have to consider that, as opposed to what we have seen in section 5.3, with constant installments where there is always a unique solution for the installment, here the variability of the installments usually leads to infinite solutions. The problem becomes determinate, and thus has a unique solution due to the linearity of the installment in (5.47) and (5.47'), only if the number of constraints between the installments at different maturities is enough to cancel out that of the degrees of freedom; i.e. *s*-1 further constraints in addition to (5.47) or (5.47'). This is obtained, in particular, imposing that the installments evolve in AP or in GP

5.5.2. *Specific cases: annuities in arithmetic progression*

Let us consider annuities in AP using

$$R_h = \ddot{R}_h = H + D\,h \tag{5.48}$$

Under the hypothesis of $R_h = \ddot{R}_h = h$ (i.e. $H=0$, $D=1$ in (5.48)) the IV of the unitary annuity in AP -immediate or -due, i.e. of the *increasing annuity* with SD law, are obtained[39] and expressed respectively by

$$I_s^{(m)} = \sum_{h=1}^{n} h\left(1 - d\,\frac{h}{m}\right) = \frac{s(s+1)}{2}\left(1 - d\,\frac{2s+1}{3m}\right) \tag{5.49}$$

$$\ddot{I}_s^{(m)} = \sum_{h=1}^{n} h\left(1 - d\,\frac{h-1}{m}\right) = \frac{s(s+1)}{2}\left(1 - d\,\frac{s^2-1}{3m}\right) \tag{5.49'}$$

Thus for the IV of annuities with delayed or advance installments given in (5.48) the following is easily obtained, from the 1st part of (5.47) and (5.47'),

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38 We only consider the IV and FV of a temporary annuity because: 1) due to the short application interval of the linear law it is not relevant to consider perpetuities; 2) for the same reason the PVDA are not important; furthermore, given the decomposability of the SD law, the PVDA is not obtained from the IV applying the discount; a direct calculation is needed.

39 We will use here: $\sum_{h=1}^{n} h^2 = \dfrac{n(n+1)(2n+1)}{6}$, $\sum_{h=1}^{n} h(h-1) = \dfrac{n(n^2-1)}{3}$.

$$V_0 = H\sum\nolimits_{h=1}^{s}\left[1-d\frac{h}{m}\right]+D\,I_s^{(m)} = s\left\{H\left[1-d\frac{s+1}{2m}\right]+D\frac{s+1}{2}\left[1-d\frac{2s+1}{3m}\right]\right\} \quad (5.50)$$

$$\ddot{V}_0 = H\sum\nolimits_{h=1}^{s}\left[1-\frac{d(h-1)}{m}\right]+D\,\ddot{I}_s^{(m)} = s\left\{H\left[1-\frac{d(s+1)}{2m}\right]+D\left[\frac{(s+1)}{2}-\frac{d}{3m}(s^2-1)\right]\right\} \quad (5.50')$$

For the FV of the aforementioned annuities, from the 2nd part of (5.47) and (5.47') the following is obtained

$$V_s = \sum\nolimits_{h=1}^{s}(H+Dh)\left\{1+i\frac{s-h}{m}\right\} = sH\left[1+\frac{i}{m}\left(s-\frac{s+1}{2}\right)\right]+D\frac{s(s+1)}{2}\left[1+i\frac{s-1}{3m}\right]$$

$$(5.51)$$

$$\ddot{V}_S = \sum\nolimits_{h=1}^{s}(H+D\,h)\left\{1+i\frac{s+1-h}{m}\right\} = sH\left[1+i\frac{s+1}{2m}\right]+D\frac{s(s+1)}{2}\left[1+i\frac{s+2}{3m}\right] \quad (5.51')$$

Exercise 5.19

Calculate the IV and FV in the -immediate and -due case, of an annuity formed by 15 monthly payments, the first one of €6,500 and the following payments varying in arithmetic progression with a ratio of €150, evaluating with linear laws and equivalent rates at the annual discount rate of 6.4%. Consider the *inverse problem* for amortization and accumulation.

A. The IV of the annuity-immediate is obtained by applying (5.50) with: H=6,350; D=150; m=12; s=15, d=0.064.

$$V_0 = 15\left\{6,350\left[1-0.064\frac{16}{24}\right]+150\frac{16}{2}\left[1-0.064\frac{31}{36}\right]\right\} = 91,186 +17,008 = 108,194.00$$

The IV of the annuity-due is obtained by applying (5.50') with: H=6,350; D=150; m=12; s=15; d=0.064. The result is:

$$\ddot{V}_0 = 15\left\{6,350\left[1-0.064\frac{16}{24}\right]+150\left[\frac{16}{2}-0.064\frac{224}{36}\right]\right\} = 91,694 +17,104 = 108,798.00$$

The FV of the annuity-immediate is obtained by applying (5.51) with: H=7,100; D=160; m=12; s=15; $i = 0.064/0.936 = 0.0683761$. The result is:

$$V_{15} = 15 \left\{ 6,350 \left[1 + \frac{0.0683761}{12} \left(15 - \frac{16}{2} \right) \right] + 150 \frac{16}{2} \left[1 + 0.0683761 \frac{14}{36} \right] \right\} = 117,527.78$$

The FV of the annuity-due is obtained by applying (5.51') with: H=7,100; D=160; m=12; s=15; $i = 0.064/0.936 = 0.068376$. The result is:

$$\ddot{V}_{15} = 15 \left\{ 6,350 \left[1 + 0.0683761 \frac{16}{24} \right] + 150 \frac{16}{2} \left[1 + 0.0683761 \frac{17}{36} \right] \right\} = 118,173.08$$

For the inverse problem, let us observe that the percentage ratio in the first installment is: $\gamma = D/(H+D) = 0.0230769$. Therefore, if we want to amortize, using an SD law, the debt of €108,194 by 15 increasing delayed monthly installments in AP at 2.30769% of the first installment, this and thus all payments are found to solve the system formed by two equations: (5.50), with the given parameters and V_0 = 108194, and $D = 0.0230769(H+D)$, in the two unknowns H and D. The result is: $H = 6350$, $D = 150$, from which the first installment is 6,500 and the other increase by 150 per month. The same installments, if paid at the beginning instead of the end of each month, are consistent to amortize a debt of €108,798, as we see using (5.50') with $\ddot{V}_0 = 108,798$.

Proceeding analogously using (5.51) and (5.51'), we can see that, with SDI law at the rate of 6.83761%, the same 15 monthly installments form a final capital of €117,527.78 if delayed, or of €118,173.08 if advance.

5.5.3. *Specific cases: annuities in geometric progression*

Let us consider the problems of section 5.5.2 using fractional annuities in GP, writing the installments in the form

$$R_h = \ddot{R}_h = R \, q^{h-1} \tag{5.52}$$

where R is the first installment, m is the frequency, s is the total number of installments and q is the ratio of the GP. Using:

$$G_s = \sum_{k=0}^{s-1} k \, q^k = \frac{(s-1)q^s}{q-1} - \frac{q^s - 1}{(q-1)^2} \tag{5.53}$$

the IV of the annuity-due with SD law at the rate d and with installments written in (5.52) is given, according to (5.53), by

$$\ddot{V}_0 = \sum_{k=0}^{S-1} R\, q^k \left(1-d\,\frac{k}{m}\right) = R\left(\frac{q^s-1}{q-1}-\frac{d}{m}G_s\right) \qquad (5.54)$$

For the IV of the annuity-immediate with installments in (5.52), for comparison with (5.54), we obtain:

$$V_0 = \sum_{h=1}^{S} R\, q^{h-1}\left(1-d\,\frac{h}{m}\right) = \ddot{V}_0 - R\,\frac{d}{m}\,\frac{q^s-1}{q-1}$$

and therefore[40]

$$V_0 = R\left\{\left(1-\frac{d}{m}\right)\frac{q^s-1}{q-1}-\frac{d}{m}G_s\right\} \qquad (5.54')$$

With similar development as above, the FV of the annuity-due with installments (5.52), according to the SDI law with rate i, is obtained. It follows that

$$\ddot{V}_S = \sum_{k=0}^{S-1} R\, q^k \left[1+\frac{i}{m}\,(s-k)\right] = R\left\{\left(1+\frac{is}{m}\right)\frac{q^s-1}{q-1}-\frac{i}{m}G_s\right\} \qquad (5.55)$$

Comparing with (5.55), we obtain

$$V_S = \sum_{h=1}^{S} R\, q^{h-1}\left(1+i\,\frac{s-h}{m}\right) = \ddot{V}_0 - R\,\frac{i}{m}\,\frac{q^s-1}{q-1}$$

and thus

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40 Observe that V_0 is the arithmetic mean of $R\dfrac{q^s-1}{q-1}$ and $-RG_s$.

$$V_S = R\left\{\left(1 + \frac{i(s-1)}{m}\right)\frac{q^s - 1}{q-1} - \frac{i}{m}G_s\right\} \tag{5.55'}$$

Exercise 5.20

A small loan is amortized over a short period according to a SD law with monthly advance installments in GP. Let us assume the following parameters:

– initial installment $R = €650$;

– variation monthly rate $= 1.2\%$;

– annual discount rate for the amortization $= 5.60\%$;

– number of monthly rate $s = 10$.

Calculate the debt to amortize and, also, the debt in the case of delayed installments.

A. Due to (5.53), (5.54) and (5.54') we have

$$G_{10} = \frac{9 \cdot 1{,}012^{10}}{0{,}012} - \frac{1{,}012^{10} - 1}{(0{,}012)^2} = 47.9724838 \ ;$$

$$\ddot{V}_0 = 650\left(9.4443164 - 0.2238716\right) = 5{,}993.29$$

$$V_0 = 650\left(0.9953333 \cdot 9.4443164 - 0.2238716\right) = 5{,}964.64$$

Obviously, if we assign, with the rate, time and ratio given above,

a) the debt of 5,993.29 to amortize with monthly advance payments;

b) the debt of 5,964.64 to amortize with monthly delayed installments,

the given installments would be found as a solution.

Exercise 5.21

An industrial company, with increasing turnover, has to replace an old plant over a short period of time. To partially finance the replacement they are able to deposit, at the beginning of every quarter, amounts increasing at 2.5%, the first of which is €6,900, into a savings account with SDIlaw at 6% per year, for 9 months. Calculate the final balance of the account. Also, calculate in the case of delayed payments.

A. $s = 3$; $q = 1.025$. Due to (5.53), (5.55) and (5.55')

$$G_3 = \frac{2 \cdot 1.025^{\,3}}{0.025} - \frac{1.025^{\,3} - 1}{0.025^{\,2}} = -36.87375$$

$$V_3 = 6900 \; (1.03 \cdot 3.0756250 + 0.5531062) = 25,674.90$$

Obviously if we assign, with the rate, time and ratio given above, the capital of 25,993.23 to accumulate with quarterly advance installments, or the capital of 25674.90 to accumulate with quarterly delayed installments, the given installments would be found as solution.